Problem Set 07: Mean Value Theorem

(1) Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be differentiable. Assume that $f(0) = g(0)$ and $f'(x) \leq g'(x)$, $\forall\, x \in \mathbb{R}$. Show that $f(x) \leq g(x)$ for $x \geq 0$.

(2) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be differentiable. Assume that $1 \leq f'(x) \leq 2$ for $x \in \mathbb{R}$ and $f(0) = 0$. Prove that $x \leq f(x) \leq 2x$ for $x \geq 0$.

(3) Use the mean value theorem (MVT) to establish the following inequalities

(a) $e^x > 1 + x$, $\forall\, x \in \mathbb{R}$.

(b) $\frac{y-x}{y} < \log \frac{y}{x} < \frac{y-x}{x}$ for $0 < x < y$.

(c) $\frac{1}{2\sqrt{n+1}} < \sqrt{n+1} - \sqrt{n} < \frac{1}{2\sqrt{n}}$, $\forall\, n \in \mathbb{N}$.

(d) If $e \leq a < b$, then $a^b > b^a$. (Hint: Use part (b)).

(e) **Bernoullis Inequality:** Let $\alpha > 0$ and $h \geq -1$. Then

$$\begin{aligned} (1+h)^\alpha &\leq 1 + \alpha h, \quad \text{for } 0 < \alpha \leq 1, \\ (1+h)^\alpha &\geq 1 + \alpha h, \quad \text{for } \alpha \geq 1. \end{aligned}$$

(4) Prove that $\frac{\sin x}{x}$ is strictly decreasing on $(0, \pi/2)$.

(5) Let $f : [0,1] \longrightarrow \mathbb{R}$ be differentiable such that $|f'(x)| < 1$, $\forall\, x \in [0,1]$. Show that f has at most one fixed point.

(6) Let $f : [0,1] \longrightarrow \mathbb{R}$ be differentiable and $f(0) = 0$. Suppose that $|f'(x)| \leq |f(x)|\ \forall\, x \in [0,1]$. Show that $f = 0$.

(7) Let $f : (0,1] \longrightarrow \mathbb{R}$ be differentiable with $|f'(x)| < 1$. Define $a_n := f(1/n)$. Show that (a_n) converges.

(8) Let $f : [0,1] \longrightarrow \mathbb{R}$ and $a_n := f(\frac{1}{n}) - f(\frac{1}{n+1})$. Prove the following:

(a) If f is continuous, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If f is differentiable and $|f'(x)| < \frac{1}{2}$, $\forall\, x \in [0,1]$, then $\sum_{n=1}^{\infty} a_n (\cos n)\sqrt{n}$ converges.

(9) Let $f : [a,b] \longrightarrow \mathbb{R}$ be differentiable and $a \geq 0$. Using Cauchy mean value theorem, show that there exist $c_1, c_2 \in (a,b)$ such that $\frac{f'(c_1)}{a+b} = \frac{f'(c_2)}{2c_2}$.

(10) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be such that $f''(c)$ exists at some $c \in \mathbb{R}$, Using L'Hopital rule, show that

$$\lim_{h \to 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2} = f''(c).$$

Show with an example that if the above limit exists then $f''(c)$ may not exists.