Problem Set 05: IVP, Existence of maxima/minima

(1) Prove that $x = \cos x$ for some $x \in (0, \pi/2)$.

(2) Prove that $xe^x = 1$ for some $x \in (0, 1)$.

(3) Are there continuous functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(x) \notin \mathbb{Q}$ for $x \in \mathbb{Q}$ and $f(x) \in \mathbb{Q}$ for $x \notin \mathbb{Q}$.

(4) Let $f : [0, 1] \longrightarrow \mathbb{R}$ be continuous. Assume that the image of f lies in $[1, 2] \cup (5, 10)$ and that $f(1/2) \in [0, 1]$. What can you conclude about the image of f.

(5) Let p be a real polynomial function of odd degree. Show that $p : \mathbb{R} \longrightarrow \mathbb{R}$ is an onto function.

(6) Let $f : [0, 2\pi] \longrightarrow [0, 2\pi]$ be continuous such that $f(0) = f(2\pi)$. Show that there exists $x \in [0, 2\pi]$ such that $f(x) = f(x + \pi)$.

(7) Show that $x^4 + 5x^3 - 7$ has at least two real roots.

(8) Let $p(X) := a_0 + a_1X + \cdots + a_nX^n$, n is even. If $a_0a_n < 0$, show that p has at least two real roots.

(9) Let $f : [a, b] \longrightarrow \mathbb{R}$ be continuous. Show that $f([a, b]) = [c, d]$ for some $c, d \in \mathbb{R}$ with $c \leq d$. Can you identify c, d.

(10) Does there exist a continuous function $f : [0, 1] \longrightarrow (0, \infty)$ which is onto.

(11) Does there exist a continuous function $f : [a, b] \longrightarrow (0, 1)$ which is onto.

(12) Let $f : [a, b] \longrightarrow \mathbb{R}$ be continuous such that $f(x) > 0$ for all $x \in [a, b]$. Show that there exists $\delta > 0$ such that $f(x) > \delta$ for all $x \in [a, b]$.

(13) Construct a continuous bijection $f : [a, b] \longrightarrow [c, d]$ such that f^{-1} is continuous.

(14) Construct a continuous function from $(0, 1)$ onto $[0, 1]$. Can such a function be one-one.

(15) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function such that

  (a) $f(\mathbb{R}) \subset (-2, -1) \cup [1, 5)$ and

  (b) $f(0) = e$.

  Can you give realistic bounds for f.