Problem Set 04: Limit and Continuity

(1) Find all values of α such that $\lim_{x\to -1} \frac{2x^2-\alpha x-14}{x^2-2x-3}$ exists. Find the limit.

(2) Let $\lim_{x\to 0} \frac{f(x)}{x^2} = 5$. Show that $\lim_{x\to 0} \frac{f(x)}{x} = 0$.

(3) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $x_0 \in \mathbb{R}$. Suppose $\lim_{x\to x_0} f(x)$ exists. Show that $\lim_{x\to 0} f(x+x_0) = \lim_{x\to x_0} f(x)$.

(4) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function such that $f(c) > 0$ for some $c \in \mathbb{R}$. Show that there exists an $\epsilon > 0$ such that $f(x) > 0$ for all $x \in (c-\epsilon, c+\epsilon)$.

(5) $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = |x|$. Show that f is continuous on $\mathbb{R}$.

(6) Discuss the continuity/discontinuity for the following functions

(a) $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = \min\{x-[x], 1+[x]-x\}$, where $[x]$ stands for the greatest integer less than or equal to x.

(b) $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} 3x+1, & \text{if } x \in \mathbb{Q} \\ x, & \text{otherwise.} \end{cases}$$

(c) $f : [0,\pi] \longrightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0, & \text{if } x = 0 \\ x\sin\frac{1}{x} - \frac{1}{x}\cos\frac{1}{x}, & \text{otherwise.} \end{cases}$$

(7) Let A be a nonempty subset of $\mathbb{R}$ and $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = \inf\{|x-a| : a \in A\}$. Show that f is continuous on $\mathbb{R}$.

(8) Let $J = \{\frac{1}{n} : n \in \mathbb{N}\}$. Show that any function $f : J \longrightarrow \mathbb{R}$ is continuous on J.

(9) A function $f : [a,b] \longrightarrow \mathbb{R}$ is said to be Lipschitz on $[a,b]$ if there exists $L > 0$ such that $|f(x) - f(y)| \leq L|x-y|$ for all $x, y \in [a,b]$. Show that any Lipschitz function is continuous.

(10) Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous such that given any two points $x < y$, there exists a point z such that $x < z < y$ and $f(z) = g(z)$. Show that $f(x) = g(x)$ for all x.

(11) Suppose $f : [0,\infty) \longrightarrow \mathbb{R}$ be continuous such that $\lim_{x\to\infty} f(x)$ exists. Show that f is bounded.

(12) Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous. If $f(x) = g(x)$ for $x \in \mathbb{Q}$, then show that $f = g$.