PROBLEM SET 02: CONVERGENCE OF SEQUENCES AND MONOTONE SEQUENCES

(1) Let (x_n) be a sequence such that $x_n \to x$ and $x_n \to y$. Prove that $x = y$.

(2) Investigate the Convergence of the Sequence (x_n) where
   (a) $x_n = \frac{\log(n)}{n}$.
   (b) $x_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} \cdots + \frac{1}{\sqrt{n^2+n}}$.
   (c) $x_n = a^n(2n)^b$ where $0 < a < 1$ and $b > 1$.
   (d) $x_n = (a^n + b^n)^{1/n}$ where $0 < a < b$.
   (e) $x_n = n^\alpha - (n+1)^\alpha$ for some $\alpha \in (0,1)$.

(3) Let $x_n = (-1)^n$. Show that the sequence (x_n) does not converge.

(4) Let (x_n) be a sequence. Prove that $x_n \to 0$ if and only if $|x_n| \to 0$.

(5) Let $x_n \to x$. Assume that $x_n \geq 0 \ \forall \ n$. Show that $x \geq 0$.

(6) Show that the sequence (x_n) is bounded and monotone, and find its limit.
   (a) $x_1 = 1$ and $x_{n+1} = \frac{4+3x_n}{3+2x_n}$.
   (b) $x_{n+1} = \frac{1}{2}(x_n + \frac{a}{x_n})$ where $a > 0$ and $x_1 > 0$. (Hint: Use AM-GM inequality.)
   (c) $x_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$.
   (d) $x_n = 1 + \frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$.

(7) Show that the sequence $x_n = 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is monotonically increasing but not bounded above. Hence, it is not convergent. Compare with Question 6 (d). Show that $x_n \to \infty$.

(8) Show that $(n!)^{\frac{1}{n}} \to \infty$.

(9) Let (x_n) be bounded. Assume that $x_{n+1} \geq x_n - 2^{-n}$. Show that (x_n) is convergent.

(10) Prove that the following sets are infinite dimensional vector spaces.
   (a) ℓ_∞: Set of all bounded sequences.
   (b) c: Set of all convergent sequences.
   (c) c_0: Set of all of all sequences converging to 0.
   (d) c_{00}: Set of all sequences whose terms are zero after some stage.
   (e) Prove $\mathrm{c}_{00} \subseteq \mathrm{c}_0 \subseteq \mathrm{c} \subseteq \ell_\infty$. Are the inlusions proper?