Problem Set 10: Series: Definition of convergence, Necessary and sufficient conditions for convergence.

(1) Prove that if a series $\sum_{n=1}^{\infty} a_n$ converges, then the sum is unique.

(2) Show that $\sum_{n=1}^{\infty} a_n$ converges if and if $\sum_{n=k}^{\infty} a_n$ converges for any $k \in \mathbb{N}$.

(3) Let (a_n) be any sequence of real numbers. Show that this sequence converges to a number S if and only if the series

$$a_1 + \sum_{n=2}^{\infty}(a_n - a_{n-1})$$

converges and has sum S. Verify the convergence/divergence of the following series:

(a) $\sum_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)}$.

(b) $\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$.

(4) Let $\sum_{n=1}^{\infty} a_n$ converges and $a_n > 0$ for all n. If (a_{n_k}) is a subsequence of (a_n), show that $\sum_{k=1}^{\infty} a_{n_k}$ also converges.

(5) Let $\sum_{n=1}^{\infty} a_n$ be a convergent series. Show that for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\sum_{n=N+1}^{\infty} a_n < \epsilon$. The series $\sum_{n=N+1}^{\infty} a_n$ is called a tail of the series $\sum_{n=1}^{\infty} a_n$.

(6) Express the infinite repeating decimal

$$0.123451234512345123451234512345\ldots$$

as the sum of a convergent geometric series and compute its sum.

(7) Show that

$$\frac{1}{r-1} = \frac{1}{r+1} + \frac{2}{r^2+1} + \frac{4}{r^4+1} + \frac{8}{r^8+1} + \cdots$$

for all $r > 1$.

(8) Obtain a formula for the following sums

(a) $2 + \dfrac{2}{\sqrt{2}} + 1 + \dfrac{1}{\sqrt{2}} + \dfrac{1}{2} + \dfrac{1}{2\sqrt{2}} + \cdots$.

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+2)(k+4)}$.

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{\alpha r + \beta}{k(k+1)(k+2)}$.

(9) Let $\sum_{n=1}^{\infty} a_n$ be a convergent series and $\sum_{n=1}^{\infty} b_n$ is obtained by grouping finite number of terms of $\sum_{n=1}^{\infty} a_n$ such as $(a_1 + a_2 + \cdots + a_{m_1}) + (a_{m_1+1} + a_{m_1+2} + \cdots + a_{m_2}) + \cdots$ for some $m_1, m_2, \ldots$ (Here $b_1 = a_1 + a_2 + \cdots + a_{m_1}, b_2 = a_{m_1+1} + a_{m_1+2} + \cdots + a_{m_2}$ and so on). Show that $\sum_{n=1}^{\infty} b_n$ converges and has the same limit as $\sum_{n=1}^{\infty} a_n$. What happens if $\sum_{n=1}^{\infty} a_n$ diverges?

(10) Let $a_n \geq 0$ for all n such that $\sum_{n=1}^{\infty} a_n$ converges. Suppose $\sum_{n=1}^{\infty} b_n$ is obtained by rearranging the terms of $\sum_{n=1}^{\infty} a_n$ (i.e., the terms of $\sum_{n=1}^{\infty} b_n$ are same as those of $\sum_{n=1}^{\infty} a_n$ but they occur in different order). Show that $\sum_{n=1}^{\infty} b_n$ converges and has the same limit as $\sum_{n=1}^{\infty} a_n$.

(11) Consider the series $\sum_{n=1}^{\infty} a_n$ where $a_n = \frac{(-1)^{n+1}}{n}$. Show that the series

$$(1 - \frac{1}{2}) - \frac{1}{4} + (\frac{1}{3} - \frac{1}{6}) - \frac{1}{8} + (\frac{1}{5} - \frac{1}{10}) - \frac{1}{12} + \cdots$$

which is obtained from $\sum_{n=1}^{\infty} a_n$ by rearranging and grouping, is $\frac{1}{2} \sum_{n=1}^{\infty} a_n$.