PROBLEM SET 01: THE REAL NUMBER SYSTEM

(1) Show that the set of natural numbers $\mathbb{N}$ is not bounded above in $\mathbb{R}$.

(2) Let A be a nonempty bounded above subset of $\mathbb{R}$. If α and β are least upper bounds of A, show that $\alpha = \beta$.

(3) Let $x, y \in \mathbb{R}$ such that $x < y$. Show that there exist $m, n \in \mathbb{N}$ such that $x < x + \frac{1}{m} < y$ and $x < y - \frac{1}{n} < y$.

(4) Find the supremum and infimum of the following sets:

(a) $\{1 - \frac{1}{n^2} : n \in \mathbb{N}\}$.

(b) $\{\frac{m}{|m|+n} : m \in \mathbb{Z},\ n \in \mathbb{N}\}$.

(c) $\{\frac{m+n}{mn} : m,\ n \in \mathbb{N}\}$.

(d) $\{x \in \mathbb{R} : x^2 - 5x + 6 < 0\}$.

(e) $\{x + x^{-1} : x > 0\}$.

(f) The set of real numbers in $(0, 1)$ whose decimal expansions contains only 0's and 1's.

(5) Let A be a nonempty subset of $\mathbb{R}$ and $\alpha \in \mathbb{R}$. Show that $\alpha = \sup A$ if and only if $\alpha - \frac{1}{n}$ not an upper bound of A but $\alpha + \frac{1}{n}$ is an upper bound of A for every $n \in \mathbb{N}$.

(6) Let A be a nonempty subset of $\mathbb{R}$ and $x \in \mathbb{R}$. Define the distance between x and A by

$$d(x, A) = \inf\{|x - a| : a \in A\}.$$

If $\alpha \in \mathbb{R}$ is the least upper bound of A, show that $d(\alpha, A) = 0$.

(7) Suppose there is an orchard, full of trees. Consider the the following statement.

In each tree in the orchard, we can find a branch on which all of the leaves are green.

Write the above statement as a mathematical statement using quantifiers. Also, write its negation.

(8) Let $A,\ B,\ C$ be three subsets of $\mathbb{R}$. What is the negation of the following statement:

For every $\epsilon > 1$, there exists $a \in A$ and $b \in B$ such that for all $c \in C$, $|a - c| < \epsilon$ and $|b - c| > \epsilon$.