On Some Classes of Projections in Banach Spaces and Related Topics

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June, 2013

On Some Classes of Projections in Banach Spaces and Related Topics

A Thesis Submitted

in Partial Fulfilment of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY

by

ABDULLAH BIN ABU BAKER

to the

Department of Mathematics and Statstics
Indian Institute of Technology, Kanpur
June, 2013

CERTIFICATE

It is certified that the work contained in the thesis titled **"On Some Classes of Projections in Banach Spaces and Related Topics"** by **Abdullah Bin Abu Baker (Roll No. Y7108061)** has been carried out under my supervision. The results presented in this thesis have not been submitted to any other university or institute for the award of any degree or diploma.

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June, 2013

Synopsis

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Degree for which submitted	:	**Ph.D.**
Department	:	**Mathematics and Statistics**
Thesis Title	:	**On Some Classes of Projections in Banach Spaces and Related Topics**
Thesis Supervisor	:	**S. Dutta**
Month and year of submission	:	**June, 2013**

In the first section, we explain the background and the main theme of this thesis and provide a chapter-wise summary of its main results. In the second section, we introduce some notation and preliminaries that will be used throughout this thesis. In the last section we state all the main results proved in this thesis.

Introduction

Projections are basic building blocks in understanding the structure of a Banach space. However, constructing a projection with desired properties often turns out to be a daunting task. By a projection we always mean a bounded linear operator P such that $P^2 = P$. We say a projection P is contractive (respectively, bi-contractive) if $\|P\| = 1$ (respectively, $\|P\| = \|I - P\| = 1$).

Attempt to describe the structure of contractive or bi-contractive projections on classical Banach spaces like $C_0(\Omega)$ or L_p and on other spaces of operators, specially C^*- algebras, had received lot of attention in past as well as in recent time. The seminal work by Lindenstrauss [27] and the book [22] by H. E. Lacey are two classical references for the study of contractive projections.

In this thesis we propose to study a class of projections which are related to the isometries. To motivate, consider an isometry T on a Banach space X such that $T^n = I$, for some $n \geq 2$. Then it is immediate that $P_0 = \frac{I+T+\cdots+T^{n-1}}{n}$ is a norm one projection on X. Also, note that (see Theorem 0.0.13 in the next section) T can always be written as $T = P_0 + \lambda_1 P_1 + \cdots + \lambda_{n-1} P_{n-1}$, where $\lambda_1, \ldots, \lambda_{n-1}$ are $(n-1)$ roots of unity and P_i, $i = 1, \ldots, n-1$ are corresponding eigen projections for T. Taking cue from above we define the following.

Definition 0.0.1. Let X be a complex Banach space. A projection P_0 on X is said to be n-circular projection, $n \geq 2$, if there exist projections $P_1, P_2, \ldots, P_{n-1}$ on X such that

(a) $P_0 \oplus P_1 \oplus \cdots \oplus P_{n-1} = I$,

(b) $P_0 + \lambda_1 P_1 + \cdots + \lambda_{n-1} P_{n-1}$ is a surjective isometry for all $\lambda_i \in \mathbb{T}$, $i = 1, 2, \ldots, n-1$.

Definition 0.0.2. Let X be a complex Banach space. A projection P_0 on X is said to be a generalized n-circular projection, $n \geq 2$, if there exist $\lambda_1, \lambda_2, \ldots, \lambda_{n-1} \in \mathbb{T} \setminus \{1\}$, λ_i, $i = 1, 2, \ldots, n-1$ which are of finite order and projections $P_1, P_2, \ldots, P_{n-1}$ on X such that

(a) $\lambda_i \neq \lambda_j$ for $i \neq j$,

(b) $P_0 \oplus P_1 \oplus \cdots \oplus P_{n-1} = I$,

(c) $P_0 + \lambda_1 P_1 + \cdots + \lambda_{n-1} P_{n-1}$ is a surjective isometry.

The case of $n = 2$ has received recent attention. A projection satisfying the condition of Definition 0.0.1 for $n = 2$ is referred as *bi-circular projection*. Similarly a projection satisfying the condition of Definition 0.0.2 for $n = 2$ is referred as *generalized bi-circular projection* (henceforth GBP). Bi-circular projections were first studied by Stachó and Zalar in [33, 34]. Their motivation to study these projections is from complex analysis, more specifically from the study of Reinhardt domains (see [32, 34]). Jamison [19] showed that bi-circular projections are Hermitian. Let $B(X)$ denote the set of all bounded linear operators on X. An operator $T \in B(X)$ is said to be Hermitian if $e^{i\theta T}$ is an isometry for every $\theta \in \mathbb{R}$. Hermitian operators on various complex Banach spaces were investigated by many authors, see for example [5], [6], [7] and [15]. As a consequence of Jamison's result, many results on bi-circular projections follow from previously established results on Hermitian operators.

The notion of generalized bi-circular projection was introduced by Fošner, Ilišević and Li in [16]. The description of generalized bi-circular projections for different Banach spaces were studied in [9, 11, 14, 16, 26]. It was shown in [26] that GBPs are bi-contractive. P. K. Lin in [26] proved that if $P + \lambda(I - P)$ is an isometry and λ is of infinite order, then P is a bi-circular projection.

The central theme of the results which we prove in this thesis is to understand the structures of GBPs and of generalized 3-circular projections in general, and in particular classical spaces like $C(\Omega)$ and spaces of matrices. It turns out that these spaces are rich with GBPs and generalized 3-circular projections.

It is quite clear from Definition 0.0.2 and the discussion presented above that the descriptions of GBPs and generalized 3-circular projections depend on the isometries under a given norm. We use results related to structures of the isometry groups on the above spaces heavily in proving our results in subsequent chapters.

We now give a chapter-wise summary of the results proved in this thesis.

In Chapter 2, we prove several results concerning the representation of projections on Banach spaces.

An operator $T \in B(X)$ is of order k (a positive integer) if $T^k = I$ and $T^i \neq I$ for any

$i < k$. A reflection is an operator of order 2. An isometric reflection is both a reflection and an isometry.

In [16], the authors show that a GBP on finite dimensional spaces with respect to various G-invariant norms is equal to the average of the identity with an isometric reflection. This result was further extended in [11] to many other spaces, for example $C(\Omega)$ and $C(\Omega, X)$. In fact it is known that, see [25, Theorem 4.4], any bi-contractive projection on $C(\Omega)$ is the average of identity and an isometric reflection. The same characterization was also proved in [10] and [21] for GBPs on spaces of Lipschitz functions, and in [26] for L_p-spaces, $1 \leq p < \infty,\ p \neq 2$.

This raises the question whether every GBP on a Banach space is equal to the average of the identity operator with an isometric reflection. In other words, whether the λ associated with a GBP is always -1. We answer this question negatively in this chapter. Further we show that if P is a GBP on X, then it is equal to the average of the identity operator and a reflection R, where R belongs to the algebra generated by the isometry associated with P. If the λ associated with P is of even order then R is an isometry, otherwise it may not be. We give an example of a P which is a GBP such that $P = \frac{I+R}{2}$, and R is not an isometry. We also give an example of a generalized 3-circular projection which is not a GBP.

Let k be a positive integer and $z = (z(0), \ldots, z(k-1))$. We define the discrete Fourier coefficient of z by $\hat{z}(m) = \sum_{n=0}^{k-1} z(n)\rho^{mn}$, where $\rho = e^{-2\pi i/k}$. Then z is the inverse discrete Fourier transform of $\hat{z}$, that is, $z = IDFT(\hat{z})$ (see [35]). If S is a subset of $\{0, \ldots, k-1\}$, we denote by δ_S the vector with components given by $\delta(i) = 1$ for $i \in S$ and $\delta(i) = 0$ otherwise.

We prove the following result.

Let $P \in B(X)$ such that $P = \lambda_0 I + \lambda_1 T + \lambda_2 T^2 + \cdots + \lambda_{n-1} T^{n-1}$, where λ_i; $i = 0, 1, \ldots, n-1$ are nonzero complex numbers and T is an operator of order n. Then P is a projection if and only if $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_{n-1})$ is the IDFT of δ_S, for some $S \subseteq \{0, \ldots, n-1\}$.

In the last section of this chapter, we extend results proved by Botelho and Jamison in [11] regarding the structure of GBPs on $C(\Omega, X)$, where Ω is a compact connected Hausdorff space and X has the Strong Banach-Stone property. We also characterize GBPs on c_0-sums of Banach spaces.

The content of this chapter is entirely taken from [1].

In Chapter 3 we describe projections in the convex hull of 3-isometries in $C(\Omega)$.

If P is a proper projection on a Banach space X which can be written as $P = \alpha T_1 + (1-\alpha)T_2$ where $T_i \in \mathcal{G}(X)$, $i = 1, 2$ and $\alpha \in (0, 1)$, then $\alpha = \frac{1}{2}$. To see this, since P is proper, there exists $0 \neq x \in X$ such that $Px = 0$. Thus, $\alpha T_1 x = -\ (1-\alpha)T_2 x$. Since T_1 and T_2 are isometries, taking norms on both sides we get $\alpha = \frac{1}{2}$. One can ask that if we take $P = \alpha_1 T_1 + \alpha_2 T_2 + \alpha_3 T_3$, where $\alpha_i > 0$, $T_i \in \mathcal{G}(C(\Omega))$; $i = 1, 2, 3$ and $\alpha_1 + \alpha_2 + \alpha_3 = 1$, whether $\alpha_i = 1/3$? In this chapter we prove that this is actually true in $C(\Omega)$.

Botelho, in [9], proved that if P is a projection which is in the convex combination of two surjective isometries on $C(\Omega)$, then P is a GBP. Here, Ω is a compact Hausdorff space

We prove that a norm one projection in the convex hull of 3 surjective isometries on $C(\Omega)$ is either a GBP or a generalized 3-circular projection. We show that, if P is a projection on $C(\Omega)$ such that $P = \alpha_1 T_1 + \alpha_2 T_2 + \alpha_3 T_3$, where $\alpha_i > 0$, $T_i \in \mathcal{G}(C(\Omega))$; $i = 1, 2, 3$ and $\alpha_1 + \alpha_2 + \alpha_3 = 1$. Then either,

(a) $\alpha_i = \frac{1}{2}$ for some $i = 1, 2, 3$ and $T_j = T_k$, $j, k \neq i$ or

(b) $\alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{3}$ and T_1, T_2, T_3 are distinct surjective isometries.

The surjective isometries on $C(\Omega)$ is given by the Banach-Stone Theorem (see Theorem 0.0.14). If $T : C(\Omega) \longrightarrow C(\Omega)$ is a surjective isometry, then there exist a homeomorphism $\phi : \Omega \longrightarrow \Omega$ and a continuous map $u : \Omega \longrightarrow \mathbb{T}$ such that

$$Tf(\omega) = u(\omega)f(\phi(\omega)), \ \ \forall\ f \in C_0(\Omega),\ \omega \in \Omega.$$

Let P_0 be a generalized 3-circular projection on X. Then as in Definition 0.0.2, we will refer to T and λ_1, λ_2, the isometry and λ_1, λ_2 associated with the generalized 3-circular projection P_0 respectively.

We also show that if P_0 is a generalized 3-circular projection on $C(\Omega)$, then λ_1, λ_2 associated with P_0 are cube roots of unity.

All the results of this chapter appeared in [2].

Chapter 4 gives complete description of generalized 3-circular projections on $\mathbb{C}^n$ with a symmetric norm and on spaces of matrices with a unitarily invariant norm and unitary congruence invariant norm.

A norm $\|\cdot\|$ on $\mathbb{C}^n$ is called symmetric if $\|\Pi x\| = \|x\|$ for all $x \in \mathbb{C}^n$ and all permutation matrices Π. A norm $\|\cdot\|$ on $\mathbb{M}_{m,n}(\mathbb{C})$ is called unitarily invariant if $\|UAV\| = \|A\|$, for all $A \in \mathbb{M}_{m,n}(\mathbb{C})$ and all unitary matrices U and V in $\mathbb{M}_m(\mathbb{C})$ and $\mathbb{M}_n(\mathbb{C})$. A unitarily invariant norm on $\mathbb{M}_{m,n}(\mathbb{C})$ is also referred as symmetric norms (see [8]). Let $S_n(\mathbb{C})$ be the set of all $n \times n$ symmetric matrices over $\mathbb{C}$. A norm $\|\cdot\|$ on $S_n(\mathbb{C})$ is called unitary congruence invariant if $\|U^tAU\| = \|A\|$ for all $A \in S_n(\mathbb{C})$, where U is any unitary matrix in $\mathbb{M}_n(\mathbb{C})$.

If P_0 is a generalized 3-circular projection $\mathbb{C}^n$ with a symmetric norm, then we show that P_0 is either a bi-circular projection or λ_1, λ_2 associated with P_0 are cube roots of unity. We actually find the complete structure of P_0.

In case of unitarily invariant norms on $\mathbb{M}_{m,n}(\mathbb{C})$, the structure of generalized 3-circular projections depends on the isometry group and on $\lambda_1 + \lambda_2$. Let $U(X)$ denotes the set of all unitary operators on a Banach space X. It is known that (see Theorem 0.0.29) if $m \neq n$, then any isometry T is of the form $T(A) = UAV$ where $U \in U(\mathbb{C}^m)$ and $V \in U(\mathbb{C}^n)$. If $m = n$, then an isometry T on $\mathbb{M}_n(\mathbb{C})$ has the form either $T(A) = UAV$ or $T(A) = UA^tV$ where U, V are unitaries in $\mathbb{M}_n(\mathbb{C})$ and A^t denotes the transpose of a matrix A.

We prove that if the isometry associated with a generalized 3-circular projection P_0 is of the form $A \longmapsto UAV$ for some $U \in U(\mathbb{C}^m)$ and $V \in U(\mathbb{C}^n)$ and $\lambda_1 + \lambda_2 = -1$, then P_0 has the form $A \longmapsto R_0AS_0 + R_1AS_1 + R_2AS_2$, where $R_i = R_i^* = R_i^2$ in $\mathbb{M}_m(\mathbb{C})$ and $S_i = S_i^* = S_i^2$ in $\mathbb{M}_n(\mathbb{C})$, $i = 0, 1, 2$.

If the isometry associated with P_0 has the same form as above and $\lambda_1 + \lambda_2 \neq -1$, then one of the following holds:

(*a*) P_0 is a bi-circular projection,

(*b*) $P_0(A) = \frac{\lambda_1 A}{2(\lambda_1 - 1)} + \frac{UAV}{1-\lambda_1^2} + \frac{\lambda_1 U^q A V^q}{2(1+\lambda_1)}$,

(c) $P_0(A) = \sum_{i=0}^{p-1} R_i A S_i$ for some $R_i = R_i^* = R_i^2$ in $\mathbb{M}_m(\mathbb{C})$ and $S_i = S_i^* = S_i^2$ in $\mathbb{M}_n(\mathbb{C})$, $i = 0, 1, \ldots, p-1$, and some $p > 3$.

If the isometry associated with P_0 has the form $A \longmapsto UA^tV$, then we get similar results as above.

The structures of generalized 3-circular projections for unitary congruence invariant norms will be also of similar nature.

The results for symmetric norms on $\mathbb{C}^n$ and on $\mathbb{M}_{m,n}(\mathbb{C})$ are from [3].

In Chapter 5 we discuss questions related to algebraic reflexivity of the set of GBPs and the set of isometries on spaces of Banach space valued continuous functions on a compact Hausdorff space.

The notion of algebraic reflexivity was first introduced in [17].

Definition 0.0.3. *Let X be a Banach space and $\mathcal{S}$ a subset of $B(X)$. The algebraic closure $\overline{\mathcal{S}}^a$ of $\mathcal{S}$ is defined to be the set*

$$\{T \in B(X) : \forall\ x \in X,\ \exists\ T_x \in \mathcal{S} \text{ such that } T(x) = T_x(x)\}.$$

$\mathcal{S}$ is said to be algebraically reflexive if $\mathcal{S} = \overline{\mathcal{S}}^a$.

Algebraic reflexivity in general and on certain class of isometries were studied by many authors, see for instance [12, 13, 14, 17, 20, 23, 29, 30, 31]. Lecture Notes by Molnar [28] gives a very comprehensive account of this theory.

For a Banach space X, let $\mathcal{G}^n(X) = \{T \in \mathcal{G}(X) : T^n = I\}$. In [14], the authors proved that for a compact Hausdorff space Ω, if $\mathcal{G}(C(\Omega))$ is algebraically reflexive, then $\mathcal{G}^2(C(\Omega))$ is also algebraically reflexive. We prove this result for vector valued continuous functions and for any $n \geq 2$.

The algebraic reflexivity of the set of generalized 3-circular projections on $C(\Omega, X)$ is still open.

Remark 0.0.4. The techniques used to describe generalized 3-circular projections in Chapter 3 and Chapter 4 can be applied to describe generalized n-circular projections as well,

$n > 3$. However, it is evident from the proofs that the number of cases occurring becomes increasingly large and difficult to handle. It seems that one needs some other approach to deal with such problems for general n.

Notation and Preliminaries

In this section, we introduce some notation and recall some definitions and results that will be used throughout this thesis.

Throughout this thesis we will assume X to be a complex Banach space. We will denote by $\mathbb{T}$, the unit circle in the complex plane.

We begin by recalling the definition of generalized n-circular projection.

Definition 0.0.5. A projection P_0 on X is said to be a generalized n-circular projection, $n \geq 2$, if there exist $\lambda_1, \lambda_2, \ldots, \lambda_{n-1} \in \mathbb{T} \setminus \{1\}$, λ_i, $i = 1, 2, \ldots, n-1$ which are of finite order and projections $P_0, P_1, \ldots, P_{n-1}$ on X such that

(a) $\lambda_i \neq \lambda_j$ for $i \neq j$,

(b) $P_0 \oplus P_1 \oplus \cdots \oplus P_{n-1} = I$,

(c) $P_0 + \lambda_1 P_1 + \cdots + \lambda_{n-1} P_{n-1}$ is a surjective isometry.

Let P_0 be a generalized n-circular projection, that is, $P_0 + \lambda_1 P_1 + \cdots + \lambda_{n-1} P_{n-1} = T$ for some surjective isometry T and λ_i, P_i are as in Definition 0.0.5, $i = 1, 2, \ldots, n-1$.

Suppose that $\lambda_m = \lambda_n$ for some m, n then we see that

$$T = P_0 + \lambda_1 P_1 + \cdots + \lambda_m (P_m + P_n) + \cdots + \lambda_{n-1} P_{n-1}.$$

As $(P_m + P_n)$ is a projection, we conclude that P_0 is a generalized $(n-1)$-circular projection.

Similarly, if $\lambda_m = -\lambda_n$, then $(P_m - P_n)$ is not a projection but $(P_m - P_n)^2 = P_m + P_n$ is. Therefore, we have

$$T = P_0 + \lambda_1 P_1 + \cdots + \lambda_m (P_m - P_n) + \cdots + \lambda_{n-1} P_{n-1}.$$

This implies that

$$T^2 = P_0 + \lambda_1^2 P_1 + \cdots + \lambda_m^2 (P_m + P_n) + \cdots + \lambda_{n-1}^2 P_{n-1}.$$

Since T^2 is an isometry, we conclude that P_0 is a generalized $(n-1)$-circular projection. So, we define the following.

Definition 0.0.6. *A generalized n-circular projection P_0 is called proper if it is not a generalized $(n-1)$-circular projection.*

Botelho in [9] introduced the notion of generalized n-circular projection as follows:

A projection P on X is said to be a generalized n-circular projection if there exists a surjective isometry T of order n such that

$$P = \frac{I + T + \cdots + T^{n-1}}{n}.$$

Remark 0.0.7. *Let $T \in \mathcal{G}(X)$ such that $T^n = I$ and $P = \frac{I+T+\cdots+T^{n-1}}{n}$. Then P is a generalized n-circular projection in the sense of Definition 0.0.5.*

To see this, we first note that P is a projection. Let $\lambda_0 = 1, \lambda_1, \ldots, \lambda_{n-1}$ be the n distinct roots of identity. For $i = 1, 2, \ldots, n-1$, we define

$$P_i = \frac{I + \overline{\lambda_i} T + \cdots + \overline{\lambda_i}^{n-1} T^{n-1}}{n}.$$

Then each P_i is a projection, $P_0 \oplus P_1 \oplus \cdots \oplus P_{n-1} = I$ and $P_0 + \lambda_1 P_1 + \cdots + \lambda_{n-1} P_{n-1} = T$.

Definition 0.0.8. *A projection P on a Banach space X is said to be bi-circular projection if $P + \lambda(I - P)$ is a surjective isometry, for all $\lambda \in \mathbb{T}$.*

Definition 0.0.9. *A projection on a Banach space X is said to be generalized bi-circular projection if there exists a $\lambda \in \mathbb{T} \setminus \{1\}$ such that $P + \lambda(I - P)$ is a surjective isometry.*

Generalized bi-circular projections are not necessarily Hermitian. If P is a GBP, then so is $I - P$.

Remark 0.0.10. *We note that in Definition 0.0.9, it is not necessary to assume that $P + \lambda(I - P)$ is surjective. It follows that this isometry is always surjective. To see this, let $x \in X$ and $y = Px + \frac{1}{\lambda}(I - P)x$. Then we have $(P + \lambda(I - P))(y) = Px + (I - P)x = x$.*

Theorem 0.0.11. *[26, Theorem 1] Let X be a complex Banach space and P a projection on X. Suppose that $P + \lambda(I - P)$ is an isometry. If λ is of infinite order in $\mathbb{T}$, then P is Hermitian.*

Definition 0.0.12. *Let P be a generalized bi-circular projection on X. The multiplicative group associated with P is defined to be the set*

$$\Lambda_P = \{\lambda \in \mathbb{T} : P + \lambda(I - P) \text{ is an isometry}\}.$$

This set is a group under multiplication.

The relation between finite order operators and projections is given in the next theorem.

Theorem 0.0.13. *Let X be a Banach space and T an operator of order n. Then there exist pairwise orthogonal projections P_i, $i = 0, 1, \ldots, n-1$ such that $T = P_0 + \lambda_1 P_1 + \cdots + \lambda_{n-1} P_{n-1}$; where $\lambda_1, \ldots, \lambda_{n-1}$ are $(n-1)$ roots of unity.*

The proof follows from [36, Theorem 5.9-E].

We denote by $C(\Omega, X)$, the space of X-valued continuous functions on compact Hausdorff space Ω while $C_0(\Omega, X)$ denotes the space of X-valued continuous functions on a locally compact Hausdorff space Ω which vanish at infinity. Both $C(\Omega, X)$ and $C_0(\Omega, X)$ are equipped with the supremum norm, that is, $\|f\|_\infty = \sup_{\omega \in \Omega} \|f(\omega)\|$. If $X = \mathbb{C}$, the above spaces are denoted by $C(\Omega)$ and $C_0(\Omega)$ respectively.

We denote by $U(X)$ and $\mathcal{G}(X)$ the group of unitary operators and surjective linear isometries on X respectively.

Theorem 0.0.14. *[4, Theorem 7.1] Let Ω be a locally compact Hausdorff space. If $T : C_0(\Omega) \longrightarrow C_0(\Omega)$ is a surjective isometry, then there exist a homeomorphism $\phi : \Omega \longrightarrow \Omega$ and a continuous map $u : \Omega \longrightarrow \mathbb{T}$ such that*

$$Tf(\omega) = u(\omega) f(\phi(\omega)), \quad \forall\ f \in C_0(\Omega),\ \omega \in \Omega.$$

For the vector-valued version of the above theorem we recall the notion of a centralizer of a Banach space, see [18, Chapter I].

Definition 0.0.15. *Let T be a bounded linear operator on a Banach space X.*

(i) *The operator T is called a multiplier of X if for every element $p \in ext(B_{X^*})$, there exists $a_T(p) \in \mathbb{C}$ such that $T^*p = a_T(p)p$. The collection of all multipliers is denoted by $Mult(X)$.*

(ii) *The centralizer of X is defined as*

$$Z(X) = \{T \in Mult(X) : \exists\, \overline{T} \in Mult(X) \text{ such that } a_{\overline{T}}(p) = \overline{a_T(p)},\ \forall\, p \in ext(B_{X^*})\}.$$

Definition 0.0.16. *A Banach space X is said to have trivial centralizer if the dimension of $Z(X)$ is equal to 1; that is, if the only elements in the centralizer are scalar multiples of the identity operator I. Obviously, this is true if X is itself the scalar field.*

Theorem 0.0.17. *[4, Theorem 8.10] Let Ω be a locally compact Hausdorff space and X a Banach with trivial centralizer. If $T : C_0(\Omega, X) \longrightarrow C_0(\Omega, X)$ is a surjective isometry, then there exist a homeomorphism $\phi : \Omega \longrightarrow \Omega$ and a map $u : \Omega \longrightarrow \mathcal{G}(X)$, continuous with respect to strong operator topology of $B(X)$, such that*

$$Tf(\omega) = u_\omega(f(\phi(\omega))), \quad \forall\ f \in C_0(\Omega),\ \omega \in \Omega.$$

For simplicity, we denote $u(\omega)$ by u_ω.

Definition 0.0.18. *[4, Definition 8.2] A Banach space X is said to have the strong Banach-Stone property if it satisfies the condition in Theorem 0.0.17.*

It is known that strictly convex spaces have trivial centralizer. In particular, they have the strong Banach-Stone property.

The following theorem describes GBPs on $C(\Omega, X)$.

Theorem 0.0.19. *[11, Theorem 2.1] If Ω is a connected compact Hausdorff space and X has the strong Banach-Stone property, then Q is a generalized bi-circular projection on $C(\Omega, X)$ if and only if one of the following statements holds:*

1. There exist a nontrivial homeomorphism $\phi : \Omega \longrightarrow \Omega$ with $\phi^2 = Id$ and a continuous function $u : \Omega \longrightarrow \mathcal{G}(X)$ with $u_\omega \circ u_{\phi(\omega)} = Id$ such that

$$Q(f)(\omega) = \frac{1}{2}[f(\omega) + u_\omega(f \circ \phi(\omega))],$$

for every $\omega \in \Omega$.

2. There exists a generalized bi-circular projection on X, P_ω, such that $Q(f)(\omega) = P_\omega(f(\omega))$, for each $\omega \in \Omega$.

Definition 0.0.20. *A projection P on a Banach space X is said to be an L_∞ projection if for every $x \in X$*

$$\|x\| = \max\{\|Px\|, \|x - Px\|\}.$$

X has trivial L_∞-structure if 0 and I are the only L_∞ projections.

Theorem 0.0.21. *[15, Theorem 2.5] Let (X_n) be a sequence of complex Banach spaces such that every X_n has trivial L_∞-structure. T is a surjective isometry of $\bigoplus_{c_0} X_n$ if and only if there exist a permutation π of $\mathbb{N}$ and a sequence of isometric operators $U_{n\pi(n)}$ such that*

$$(Tx)_n = U_{n\pi(n)} x_{\pi(n)} \text{ for each } x = (x_n) \in \bigoplus_{c_0} X_n.$$

Moreover, the space $X_{\pi(n)} \cong X_n$.

We now recall some definitions and remarks which will be used in Chapter 4.

Definition 0.0.22. *Let X be a Banach space and G a closed subgroup of $\mathcal{G}(X)$. A norm $\|\cdot\|$ on X is said to be G-invariant if*

$$\|g(x)\| = \|x\| \quad \forall\, g \in G,\ x \in X.$$

Trivially, multiple of the inner product norm on $\mathbb{C}^n$ is G-invariant.

The following theorem shows that GBPs on finite dimensional Banach spaces are orthogonal projections.

Theorem 0.0.23. *[16, Proposition 2.1] Let X be an n-dimensional inner product space and $\|\cdot\|$ a multiple of the norm induced by the inner product. Suppose $P : X \longrightarrow X$ is a non-trivial linear projection and $\lambda \in \mathbb{T} \setminus \{1\}$. The following conditions are equivalent:*

(i) $P + \lambda(I - P)$ is an isometry,

(ii) P is an orthogonal projection, that is, there exists an orthonormal basis $\{e_1, \ldots, e_n\}$ for X such that $P(e_j) = \lambda_j e_j$ where $\lambda_j \in \{0, 1\}$ for all $j = 1, \ldots, n$.

Remark 0.0.24. In the sequel, we will prove our results for G-invariant norms which are not multiple of the inner product norm.

Definition 0.0.25. *A square matrix P is called a permutation matrix if exactly one entry in each row and column is equal to 1, and all other entries are 0.*

Every permutation matrix corresponds to a unique permutation. A permutation matrix will always be in the form

$$\begin{bmatrix} e_{a_1} \\ e_{a_2} \\ \vdots \\ e_{a_n} \end{bmatrix}$$

where e_{a_j} denotes a row vector of length n with 1 in the j^{th} position and 0 in every other position and

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ a_1 & a_1 & \cdots & a_n \end{pmatrix}$$

is the corresponding permutation form of the permutation matrix.

Definition 0.0.26. *A norm $\|\cdot\|$ on $\mathbb{C}^n$ is called symmetric if $\|\Pi x\| = \|x\|$ for all $x \in \mathbb{C}^n$ and all permutation matrices Π.*

Let G be the group of all generalized permutation matrices, that is, matrices of the form DP where D is a diagonal matrix with all its elements of unit modulus and P is a permutation matrix.

The isometry group of a given symmetric norm is characterized in the following theorem.

Theorem 0.0.27. *[24, Theorem 2.5] The isometry group of a symmetric norm on* $\mathbb{C}^n$ *is* G.

Definition 0.0.28. *A norm* $\|\cdot\|$ *on* $\mathbb{M}_{m,n}(\mathbb{C})$ *is called unitarily invariant if* $\|UAV\| = \|A\|$, *for all* $A \in \mathbb{M}_{m,n}(\mathbb{C})$ *and all unitary matrices* U *and* V *in* $\mathbb{M}_m(\mathbb{C})$ *and* $\mathbb{M}_n(\mathbb{C})$ *respectively.*

Let G be the group of all linear operators on $\mathbb{M}_{m,n}(\mathbb{C})$ of the form $A \longmapsto UAV$ for some fixed unitary $U \in \mathbb{M}_m(\mathbb{C})$ and $V \in \mathbb{M}_n(\mathbb{C})$.

We denote by τ the transposition operator on $\mathbb{M}_n(\mathbb{C})$, that is, $\tau(A) = A^t$.

The isometry group of a unitarily invariant norm is described in the following theorem by Li.

Theorem 0.0.29. *[24, Theorem 2.4] The isometry group of a unitarily invariant norm* $\|\cdot\|$ *on* $\mathbb{M}_{m,n}(\mathbb{C})$ *must be of one of the following forms:*

(a) *If* $m \neq n$, $\mathcal{G}(X) = G$;

(b) *If* $m = n$, $\mathcal{G}(X) = \langle G, \tau \rangle$.

The following proposition will be used in Chapter 4.

Proposition 0.0.30. *[16, Proposition 4.1] Let* $\|\cdot\|$ *be a unitarily invariant norm on* $\mathbb{M}_{m,n}(\mathbb{C})$ *not equal to a multiple of the Frobenius norm, and* $\mathcal{K}$ *the isometry group of* $\|\cdot\|$. *Suppose* $P : \mathbb{M}_{m,n}(\mathbb{C}) \longrightarrow \mathbb{M}_{m,n}(\mathbb{C})$ *is a non-trivial linear projection and* $\lambda \in \mathbb{T}\setminus\{1\}$. *Then* $P + \lambda(I - P) \in \mathcal{K}$ *if and only if one of the following holds:*

(a) *There exists* $R \in \mathbb{M}_m(\mathbb{C})$ *with* $R = R^* = R^2$ *such that* P *has the form* $A \longmapsto RA$ *or there exists* $S \in \mathbb{M}_n(\mathbb{C})$ *with* $S = S^* = S^2$ *such that* P *has the form* $A \longmapsto AS$.

(b) $\lambda = -1$, *and there exist* $R = R^* = R^2$ *in* $\mathbb{M}_m(\mathbb{C})$ *and* $S = S^* = S^2$ *in* $\mathbb{M}_n(\mathbb{C})$ *such that* P *has the form* $A \longmapsto RAS + (I_m - R)A(I_n - S)$.

(c) $m = n$, $\lambda = -1$, *and there is* $U \in U(\mathbb{C}^n)$ *such that* P *or* $\overline{P}$ *has the form* $A \longmapsto (A + UA^t\overline{A})/2$.

Definition 0.0.31. *A norm $\|\cdot\|$ on $S_n(\mathbb{C})$, the space of all $n \times n$ symmetric matrices over $\mathbb{C}$, is called unitary congruence invariant if $\|U^tAU\| = \|A\|$ for all $A \in S_n(\mathbb{C})$, where U is an any unitary matrix in $\mathbb{M}_n(\mathbb{C})$.*

Let G be the group of all linear operators on $S_n(\mathbb{C})$ of the form $A \longmapsto U^tAU$ for some fixed unitary $U \in \mathbb{M}_n(\mathbb{C})$.

The isometry group of a unitary congruence invariant norm on $S_n(\mathbb{C})$ is described in the following theorem.

Theorem 0.0.32. *[24, Theorem 2.8] The isometry group of a unitary congruence invariant norm on $S_n(\mathbb{C})$ which is not a multiple of the Frobenius norm is G.*

The following proposition will be used in Chapter 4.

Proposition 0.0.33. *[16, Proposition 5.1] Let $\|\cdot\|$ be a unitary congruence invariant norm on $S_n(\mathbb{C})$, which is not a multiple of the Frobenius norm, and $\mathcal{K}$ the isometry group of $\|\cdot\|$. Suppose $P : S_n(\mathbb{C}) \longrightarrow S_n(\mathbb{C})$ is a non-trivial linear projection and $\lambda \in \mathbb{T} \setminus \{1\}$. Then $P + \lambda(I - P) \in \mathcal{K}$ if and only if $\lambda = -1$ and there exists $R = R^* = R^2$ in $\mathbb{M}_n(\mathbb{C})$ such that P or $\overline{P}$ has the form $A \longmapsto R^tAR + (I - R^t)A(I - R)$.*

The next theorem gives sufficient condition regarding the algebraic reflexivity of the set of isometric reflections on $C(\Omega)$.

Theorem 0.0.34. *[14, Theorem 1] Let Ω be compact Hausdorff space. If $\mathcal{G}(C(\Omega))$ is algebraically reflexive, then $\mathcal{G}^2(C(\Omega))$ is also algebraically reflexive.*

The following theorem gives conditions on X so that $\mathcal{G}(C(\Omega, X))$ is is algebraically reflexive.

Theorem 0.0.35. *[20, Theorem 7] Suppose Ω is a first countable compact Hausdorff space and X a uniformly convex Banach space such that $\mathcal{G}(X)$ is algebraically reflexive . Then $\mathcal{G}(C(\Omega, X))$ is algebraically reflexive.*

Statement of Theorems

In this section we give chapter-wise statement of all the main results proved in this thesis.

CHAPTER 2

Theorem 0.0.36. *Let X be a Banach space. If P is a projection such that $P+\lambda(I-P)=T$, where $\lambda \in \mathbb{T}\setminus\{1\}$ and T is an isometry on X, then $R=2P-I$ belongs to the algebra generated by T.*

Theorem 0.0.37. *Let P a bounded operator on a Banach space X. Let $\lambda_0,\dots,\ \lambda_{k-1}$ be nonzero complex numbers and $P=\sum_{i=0}^{k-1}\lambda_i T^i$, where T is an operator of order k. Then P is a projection if and only if $\lambda=(\lambda_0,\ \lambda_1,\ \dots,\lambda_{k-1})$ is the IDFT of δ_S, for some $S\subseteq\{0,\dots,k-1\}$.*

Theorem 0.0.38. *Let Ω be a locally compact Hausdorff space and X a Banach space with trivial centralizer. Let P be a GBP on $C_0(\Omega,X)$. Then one and only one of the following assertions holds.*

(a) *$P=\frac{I+T}{2}$, where T is an isometry on $C_0(\Omega,X)$.*

(b) *$Pf(\omega)=P_\omega(f(\omega))$, where P_ω is a generalized bi-circular projection on X.*

CHAPTER 3

Theorem 0.0.39. *Let Ω be a compact connected Hausdorff space and P_0 a proper generalized 3-circular projection on $C(\Omega)$. Then there exists a surjective isometry T on $C(\Omega)$ such that*

(a) *$P_0+\omega P_1+\omega^2P_2=T$, where P_1 and P_2 are as in Definition 0.0.5 and ω is a cube root of unity,*

(b) *$T^3=I$.*

Hence, $P_0=\frac{I+T+T^2}{3}$.

Theorem 0.0.40. *Let Ω be a compact connected Hausdorff space. Let P be a projection on $C(\Omega)$ such that $P = \alpha_1 T_1 + \alpha_2 T_2 + \alpha_3 T_3$, where $T_1,\ T_2,\ T_3$ are surjective isometries of $C(\Omega)$, $\alpha_i > 0,\ i = 1, 2, 3$ and $\alpha_1 + \alpha_2 + \alpha_3 = 1$. Then either,*

(a) *$\alpha_i = \frac{1}{2}$ for some $i = 1, 2, 3$ $\alpha_j + \alpha_k = \frac{1}{2},\ j, k \neq i$ and $T_j = T_k$ or*

(b) *$\alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{3}$ and $T_1,\ T_2,\ T_3$ are distinct surjective isometries. Moreover, in this case there exists a surjective isometry T on $C(\Omega)$ such that $T^3 = I$ and $P = \frac{I+T+T^2}{3}$.*

CHAPTER 4

Theorem 0.0.41. *Let $\|\cdot\|$ be a symmetric norm on $\mathbb{C}^n$ and P_0 a generalized 3-circular projection. Then one and only one of the following assertions holds:*

(a) *P_0 is a bi-circular projection.*

(b) *There exist $m \geq 0,\ k \geq 1$, projections $P_{0,i},\ i = 0, \ldots, k$ such that P_0 is permutationally similar to $P_{0,1} \oplus P_{0,2} \oplus \cdots \oplus P_{0,k} \oplus P_{0,0}$, where*

$$P_{0,i} = \frac{1}{3}\begin{pmatrix} 1 & d_{i1} & d_{i1}d_{i2} \\ d_{i2}d_{i3} & 1 & d_{i2} \\ d_{i3} & d_{i1}d_{i3} & 1 \end{pmatrix} \text{ and } P_{0,0} = diag(p_1, p_2, \ldots, p_m)$$

with $p_j \in \{0, 1\}$ for all $j = 1, 2, \ldots, m$ and $d_{i1}d_{i2}d_{i3} = 1$.

Theorem 0.0.42. *Let $\|\cdot\|$ be a unitarily invariant norm on $\mathbb{M}_{m,n}(\mathbb{C})$ and P_0 a generalized 3-circular projection such that the isometry associated with it is of the form $A \longmapsto UAV$ for some $U \in U(\mathbb{C}^m)$ and $V \in U(\mathbb{C}^n)$. Suppose $\lambda_1 + \lambda_2 = -1$, then there exist $R_i = R_i^* = R_i^2$ in $\mathbb{M}_m(\mathbb{C})$ and $S_i = S_i^* = S_i^2$ in $\mathbb{M}_n(\mathbb{C})$, $i = 0, 1, 2$ such that P_0 has the form $A \longmapsto R_0AS_0 + R_1AS_1 + R_2AS_2$.*

Theorem 0.0.43. *Let $\|\cdot\|$ be a unitarily invariant norm on $\mathbb{M}_{m,n}(\mathbb{C})$ and P_0 a generalized 3-circular projection such that the isometry associated with P_0 is of the form $A \longmapsto UAV$ for some $U \in U(\mathbb{C}^m)$ and $V \in U(\mathbb{C}^n)$. Suppose $\lambda_1 + \lambda_2 \neq -1$, then one and only one of following assertions holds:*

(a) *There exists $R \in \mathbb{M}_n(\mathbb{C})$ with $R = R^* = R^2$ such that $P_0(A) = AR$, or there exists $S \in \mathbb{M}_m(\mathbb{C})$ with $S = S^* = S^2$ such that $P_0(A) = SA$.*

In both cases, P_0 is a bi-circular projection.

(b) *$\lambda_i^2 = \lambda_j$, $i, j = 1, 2$ and $i \neq j$;*

(b1) *λ_1 is of order p and λ_2 is of order q with $p = 2q$. In this case we have one of the following conditions:*

(i) *P_0 is a bi-circular projection.*

(ii) *P_1 is generalized bi-circular projection and $(\lambda_1)^{p/2} = (\lambda_2)^{q/2} = -1$. Moreover, P_0 has the form*

$$A \longmapsto \frac{\lambda_1 A}{2(\lambda_1 - 1)} + \frac{UAV}{1 - \lambda_1^2} + \frac{\lambda_1 U^q A V^q}{2(1 + \lambda_1)}.$$

(b2) *$\lambda_i = \sqrt{\lambda_j}$ and λ_1, λ_2 are of order p, where p is an odd integer greater or equal to 5. Moreover, there exist $R_i = R_i^* = R_i^2$ in $\mathbb{M}_m(\mathbb{C})$ and $S_i = S_i^* = S_i^2$ in $\mathbb{M}_n(\mathbb{C})$ such that*

$$P_0(A) = \sum_{i=0}^{p-1} R_i A S_i,$$

where $i = 0, 1, \ldots, p-1$.

(c) *$\lambda_1\lambda_2 = 1$ and P_0 will have the same form as in (b2).*

Theorem 0.0.44. *Let $\|\cdot\|$ be a unitarily invariant norm on $\mathbb{M}_n(\mathbb{C})$ and P_0 a generalized 3-circular projection such that the isometry associated with P_0 is of the form $A \longmapsto UA^tV$ for some U, $V \in U(\mathbb{C}^n)$. Then one and only one of the following assertions holds:*

(a) *$\lambda_1^2 + \lambda_2^2 = -1$ and there exist $R_i = R_i^* = R_i^2$ and $S_i = S_i^* = S_i^2$ in $\mathbb{M}_n(\mathbb{C})$, $i = 0, 1, 2$ such that P_0 has the form $A \longmapsto R_0AS_0 + R_1AS_1 + R_2AS_2$.*

(b) $\lambda_i^4 = \lambda_j^2$, *$i, j = 1, 2$ and $i \neq j$;*

(b1) λ_1^2 is of order p and λ_2^2 is of order q with $p = 2q$. In this case, we have one of the following conditions:

(i) P_0 is a bi-circular projection.

(ii) P_1 is generalized bi-circular projection and $\lambda_1^p = \lambda_2^q = -1$. Moreover, P_0 has the form

$$A \longmapsto \frac{\lambda_1^2 A}{2(\lambda_1^2 - 1)} + \frac{UV^tAU^tV}{1 - \lambda_1^4} + \frac{\lambda_1^2 (UV^t)^q A (U^tV)^q}{2(1 + \lambda_1^2)}.$$

(b2) $\lambda_i^2 = \lambda_j$; λ_1^2 and λ_2^2 are of order p, where p is an odd integer greater or equal to 5. Moreover, there exist $R_i = R_i^* = R_i^2$ and $S_i = S_i^* = S_i^2$ in $\mathbb{M}_n(\mathbb{C})$ such that

$$P_0(A) = \sum_{i=0}^{p-1} R_i A S_i,$$

where $i = 0, 1, \ldots, p - 1$.

Theorem 0.0.45. *Let $\|\cdot\|$ be a unitary congruence invariant norm on $S_n(\mathbb{C})$ and P_0 a generalized 3-circular projection. Then there exists $U \in U(\mathbb{C}^n)$ such that one and only one of the following assertions holds:*

(a) U has three distinct eigenvalues. In this case, $\lambda_1 + \lambda_2 = -1$. Moreover, there exists $R_i = R_i^* = R_i^2$ in $\mathbb{M}_n(\mathbb{C})$ such that P_0 has the form $A \longmapsto R_0^t A R_0 + R_1^t A R_2 + R_2^t A R_1$.

(b) U has two distinct eigenvalues. In this case, one and only one of the following occurs:

(b1) $\lambda_i = \sqrt{\lambda_j}$, $i, j = 1, 2$ and $i \neq j$ and λ_i's are of order p, where p is an odd integer greater or equal to 3. Moreover, there exist $R_i = R_i^* = R_i^2$ and $S_i = S_i^* = S_i^2$ in $\mathbb{M}_n(\mathbb{C})$ such that

$$P_0(A) = \sum_{i=0}^{p-1} R_i A S_i,$$

where $i = 0, 1, \ldots, p - 1$.

(b2) $\lambda_1 \lambda_2 = 1$ and P_0 will have the same form as in (b1).

CHAPTER 5

Theorem 0.0.46. *Let Ω be a locally compact Hausdorff space and X a Banach space with trivial centralizer. If $\mathcal{G}(C_0(\Omega, X))$ is algebraically reflexive, then $\mathcal{G}^n(C_0(\Omega, X))$ is algebraically reflexive.*

Corollary 0.0.47. *Let Ω be a first countable compact Hausdorff space and X a uniformly convex Banach space such that $\mathcal{G}(X)$ is algebraically reflexive. Furthermore, assume that X does not have any generalized bi-circular projections. Then the set of generalized bi-circular projections on $C(\Omega, X)$ is algebraically reflexive.*

Dedicated

to

My Family

Acknowledgements

Completing my PhD perhaps has been the most challenging task in my life. I shared the best and worst moments of my doctoral journey with many people. It has been a privilege to spend several years in the Department of Mathematics, IIT Kanpur and its members will always remain dear to me.

The list of the people I need to thank would not fit in a single Acknowledgement. I just mention some people whose contributions are significant.

I take immense pleasure to express my deep sense of gratitude to my supervisor Dr. S Dutta for his guidance and support throughout my doctoral research. I am grateful to him for his encouragement and active participation. It has certainly been a great learning experience for me on both professional and personal fronts and words cannot truly express my deepest gratitude and appreciation for him.

Professor Fernanda Botelho and Professor James Jamison, I thank both of you from the bottom of my heart for all your help and especially your kind hospitality during my visit to the University of Memphis, TN, USA. Our joint work constitutes an important part of my research and I thank you for permitting me to include it in my thesis. I am honored to be associated with you and hope to continue our collaboration in future.

I offer my gratitude to FB 551 where two-thirds of my research was done. I am very grateful to that room and to the facilities it provided me for a period two years. Apart from this, I always found support, motivation, inspiration and solutions to almost all problems after walking into that room. I will perhaps never forget FB 551 ever.

I would like to express my sincere thanks to Professor P. Shunmugaraj for his help, support and guidance throughout my stay at IIT Kanpur.

I acknowledge Professor T. S. S. R. K. Rao of ISI Bangalore with whom I had several interactions and enjoyed many fruitful discussions.

My deepest gratitude to Professors Shobha Madan, A. K. Maloo, G. Santhanam, Parasar Mohanty, Sameer Chavan and Nandini Nilakantan, who provided me with several guidelines and brain storming sessions. Their unconditional support is unforgettable.

A good support system is important to survive and staying sane. I was lucky to be part Department of Mathematics at IIT Kanpur. I take this opportunity to thank Professor Neeraj Misra and Professor Debasis Kundu who always have been encouraging and accommodative. I also also extend my thanks to all department staff for their cooperation during this period.

I am deeply indebted to Professor Harish Karnick who provided me with mentorship while I was a resident of Hall-4. I also thank Professors Partha Chakroborty, A. K. Ghosh, Javed Malik, Sudhir Misra, Jaleel Akhtar, Rahul Varman and Shakeel sir for their encouragement. Dr. Sharif was like a brother to me and helped me through my tough times at IIT Kanpur.

I could not have done all these if Ambuj and Binoy were not there with me during the difficult times of my life. The 11 A.M discussions in the chemical canteen literally changed my life. I am very lucky to have them as my friends.

Pratyoosh bhai, I am very much obliged to you for your help and support, especially the time you have given me during my initial stages of PhD.

I have no words to describe Vivek Kumar Mehta. Getting to know him was the best thing that happened to me in IIT Kanpur.

I thank all my friends for their companionship and I will cherish their memories for long. Special thanks to Abdur Raheem, Abrar bhai, Adham Mustafa, akhlaq bhai, Alpesh, Anuradha, Aquil bhai, Arbaz, Arshad, Arun, Asfar, Ashiq, Awanish, Ayaz, Danish Lohani, Ehtesham, Gufran, Ibrahim, Jagabandhu, Javid bhai, Kashif, Khaleeq bhai, Lokpati, Maqbal, Mazhar, Musheer, Nadeem Akhtar, Nishat, Parvez, Peeyush, Purnendu, Quaff bhai, Rameez, Rashid, Ravi, Sachin Singh Gautam, Sabiha, Saiful Islam, Sajjad bhai, Sameer Ahmed, saurabh, Shameem, Shariq, Shiblee bhai, Suddhasheel, Sudheendra Rao, Sumit, Surjit, Surjit Kaur, Taseen, Tony Jacob, Waqur, Yousuf bhai, Zaid, Zubair.

My sincere thanks also goes to my best friend Zegham who always encouraged and believed in me.

I would thank my family from the bottom of my heart but for them my heart has no bottom. This thesis is dedicated to them.

A person who I always looked up to for inspiration, advice and guidance is my elder brother Mohammed. I can't express the affection I have for his wife Madeeha and his daughter Fatima.

A special thanks to my brothers, Obaidullah, Usman and Omer. My little sister Ayesha and her daughter Khadeeja are the biggest joys of my life. Their naughtiness playfulness flushed away every problem in my life.

None of this could have happened without the love and support of my wife Kulsum. Her patience and unwavering love were undeniably the bedrock upon which the past three years of my life have been built. Her tolerance of my occasional mood swings is a testament in itself of her unyielding devotion and love.

Many thanks to all my relatives, especially, Dr. Ashraf Hussain, Dr. Jamil Ansari, Jugnu phuppi, Nannu chacha, Nishat Khala, Suroor mamu and my in-laws for their unconditional love.

I would not be where I am today if not for my parents who sacrificed their time and energy for me. I am forever in their debt and I hope this thesis is a worthy reflection of their commitment. They have always given me the strength and wisdom to be sincere in my work, setting high moral standards and supporting me through their hard work.

Most importantly, I thank Almighty Allah for the uncountable blessings and grants that He bestowed on me. I thank Him for all the opportunities and strength showered on me in my life.

June, 2013 **Abdullah Bin Abu Baker**

Foreword

Contents

CHAPTER 1

Introduction

In the first part of this chapter, we explain the background and the main theme of this thesis and provide a chapter-wise summary of its main results. In the second part, we introduce some notation and preliminaries that will be used throughout this thesis.

1.1 Introduction

Projections are basic building blocks in understanding the structure of a Banach space. However, constructing a projection with desired properties often turns out to be a daunting task. By a projection we always mean a bounded linear operator P such that $P^2 = P$. We say a projection P is contractive (respectively, bi-contractive) if $\|P\| = 1$ (respectively, $\|P\| = \|I - P\| = 1$).

Attempt to describe the structure of contractive or bi-contractive projections on classical Banach spaces like $C_0(\Omega)$ or L_p and on other spaces of operators, specially C^*- algebras, had received lot of attention in past as well as in recent time. The seminal work by Lindenstrauss [27] and the book [22] by H. E. Lacey are two classical references for the study of contractive projections.

In this thesis we propose to study a class of projections which are related to the isometries. To motivate, consider an isometry T on a Banach space X such that $T^n = I$, for some $n \geq 2$. Then it is immediate that $P_0 = \frac{I+T+\cdots+T^{n-1}}{n}$ is a norm one projection on X. Also, note that (see Theorem 1.2.9 in the next section) T can always be written as $T = P_0 + \lambda_1 P_1 + \cdots + \lambda_{n-1} P_{n-1}$, where $\lambda_1, \ldots, \lambda_{n-1}$ are $(n-1)$ roots of unity and P_i, $i = 1, \ldots, n-1$ are corresponding eigen projections for T. Taking cue from above we define the following.

Definition 1.1.1. Let X be a complex Banach space. A projection P_0 on X is said to be n-circular projection, $n \geq 2$, if there exist projections $P_1, P_2, \ldots, P_{n-1}$ on X such that

(a) $P_0 \oplus P_1 \oplus \cdots \oplus P_{n-1} = I$,

(b) $P_0 + \lambda_1 P_1 + \cdots + \lambda_{n-1} P_{n-1}$ is a surjective isometry for all $\lambda_i \in \mathbb{T}$, $i = 1, 2, \ldots, n-1$.

Definition 1.1.2. Let X be a complex Banach space. A projection P_0 on X is said to be a generalized n-circular projection, $n \geq 2$, if there exist $\lambda_1, \lambda_2, \ldots, \lambda_{n-1} \in \mathbb{T} \setminus \{1\}$, λ_i, $i = 1, 2, \ldots, n-1$ which are of finite order and projections $P_1, P_2, \ldots, P_{n-1}$ on X such that

(a) $\lambda_i \neq \lambda_j$ for $i \neq j$,

(b) $P_0 \oplus P_1 \oplus \cdots \oplus P_{n-1} = I$,

(c) $P_0 + \lambda_1 P_1 + \cdots + \lambda_{n-1} P_{n-1}$ is a surjective isometry.

The case of $n = 2$ has received recent attention. A projection satisfying the condition of Definition 1.1.1 for $n = 2$ is referred as *bi-circular projection*. Similarly a projection satisfying the condition of Definition 1.1.2 for $n = 2$ is referred as *generalized bi-circular projection* (henceforth GBP). Bi-circular projections were first studied by Stachó and Zalar in [33, 34]. Their motivation to study these projections is from complex analysis, more specifically from the study of Reinhardt domains (see [32, 34]). Jamison [19] showed that bi-circular projections are Hermitian. Let $B(X)$ denote the set of all bounded linear operators on X. An operator $T \in B(X)$ is said to be Hermitian if $e^{i\theta T}$ is an isometry for every $\theta \in \mathbb{R}$. Hermitian operators on various complex Banach spaces were investigated by many authors, see for example [5], [6], [7] and [15]. As a consequence of Jamison's result, many results on bi-circular projections follow from previously established results on Hermitian operators.

The notion of generalized bi-circular projection was introduced by Fošner, Ilišević and Li in [16]. The description of generalized bi-circular projections for different Banach spaces were studied in [9, 11, 14, 16, 26]. It was shown in [26] that GBPs are bi-contractive. P. K. Lin in [26] proved that if $P + \lambda(I - P)$ is an isometry and λ is of infinite order, then P is a bi-circular projection.

The central theme of the results which we prove in this thesis is to understand the structures of GBPs and of generalized 3-circular projections in general, and in particular classical spaces like $C(\Omega)$ and spaces of matrices. It turns out that these spaces are rich with GBPs and generalized 3-circular projections.

It is quite clear from Definition 1.1.2 and the discussion presented above that the descriptions of GBPs and generalized 3-circular projections depend on the isometries under a given norm. We use results related to structures of the isometry groups on the above spaces heavily in proving our results in subsequent chapters.

We now give a chapter-wise summary of the results proved in this thesis.

In Chapter 2, we prove several results concerning the representation of projections on Banach spaces.

An operator $T \in B(X)$ is of order k (a positive integer) if $T^k = I$ and $T^i \neq I$ for any $i < k$. A reflection is an operator of order 2. An isometric reflection is both a reflection and an isometry.

In [16], the authors show that a GBP on finite dimensional spaces with respect to various G-invariant norms is equal to the average of the identity with an isometric reflection. This result was further extended in [11] to many other spaces, for example $C(\Omega)$ and $C(\Omega, X)$. In fact it is known that, see [25, Theorem 4.4], any bi-contractive projection on $C(\Omega)$ is the average of identity and an isometric reflection. The same characterization was also proved in [10] and [21] for GBPs on spaces of Lipschitz functions, and in [26] for L_p-spaces, $1 \leq p < \infty,\ p \neq 2$.

This raises the question whether every GBP on a Banach space is equal to the average of the identity operator with an isometric reflection. In other words, whether the λ associated with a GBP is always -1. We answer this question negatively in this chapter. Further we show that if P is a GBP on X, then it is equal to the average of the identity operator and a reflection R, where R belongs to the algebra generated by the isometry associated with P. If the λ associated with P is of even order then R is an isometry, otherwise it may not be. We give an example of a P which is a GBP such that $P = \frac{I+R}{2}$, and R is not an isometry. We also give an example of a generalized 3-circular projection which is not a GBP.

Let k be a positive integer and $z = (z(0), \ldots, z(k-1))$. We define the discrete Fourier coefficient of z by $\hat{z}(m) = \sum_{n=0}^{k-1} z(n)\rho^{mn}$, where $\rho = e^{-2\pi i/k}$. Then z is the inverse discrete Fourier transform of $\hat{z}$, that is, $z = IDFT(\hat{z})$ (see [35]). If S is a subset of $\{0, \ldots, k-1\}$, we denote by δ_S the vector with components given by $\delta(i) = 1$ for $i \in S$ and $\delta(i) = 0$ otherwise.

We prove the following result.

Let $P \in B(X)$ such that $P = \lambda_0 I + \lambda_1 T + \lambda_2 T^2 + \cdots + \lambda_{n-1} T^{n-1}$, where λ_i; $i = 0, 1, \ldots, n-1$ are nonzero complex numbers and T is an operator of order n. Then P is a projection if and only if $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_{n-1})$ is the IDFT of δ_S, for some $S \subseteq \{0, \ldots, n-1\}$.

In the last section of this chapter, we extend results proved by Botelho and Jamison in [11] regarding the structure of GBPs on $C(\Omega, X)$, where Ω is a compact connected Hausdorff space and X has the Strong Banach-Stone property. We also characterize GBPs on c_0-sums of Banach spaces.

The content of this chapter is entirely taken from [1].

In Chapter 3 we describe projections in the convex hull of 3-isometries in $C(\Omega)$.

If P is a proper projection on a Banach space X which can be written as $P = \alpha T_1 + (1-\alpha)T_2$ where $T_i \in \mathcal{G}(X)$, $i = 1, 2$ and $\alpha \in (0, 1)$, then $\alpha = \frac{1}{2}$. To see this, since P is proper, there exists $0 \neq x \in X$ such that $Px = 0$. Thus, $\alpha T_1 x = -\,(1-\alpha)T_2 x$. Since T_1 and T_2 are isometries, taking norms on both sides we get $\alpha = \frac{1}{2}$. One can ask that if we take $P = \alpha_1 T_1 + \alpha_2 T_2 + \alpha_3 T_3$, where $\alpha_i > 0$, $T_i \in \mathcal{G}(C(\Omega))$; $i = 1, 2, 3$ and $\alpha_1 + \alpha_2 + \alpha_3 = 1$, whether $\alpha_i = 1/3$? In this chapter we prove that this is actually true in $C(\Omega)$.

Botelho, in [9], proved that if P is a projection which is in the convex combination of two surjective isometries on $C(\Omega)$, then P is a GBP. Here, Ω is a compact Hausdorff space

We prove that a norm one projection in the convex hull of 3 surjective isometries on $C(\Omega)$ is either a GBP or a generalized 3-circular projection. We show that, if P is a projection on $C(\Omega)$ such that $P = \alpha_1 T_1 + \alpha_2 T_2 + \alpha_3 T_3$, where $\alpha_i > 0$, $T_i \in \mathcal{G}(C(\Omega))$; $i = 1, 2, 3$ and $\alpha_1 + \alpha_2 + \alpha_3 = 1$. Then either,

(a) $\alpha_i = \frac{1}{2}$ for some $i = 1, 2, 3$ and $T_j = T_k$, $j, k \neq i$ or

(b) $\alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{3}$ and T_1, T_2, T_3 are distinct surjective isometries.

The surjective isometries on $C(\Omega)$ is given by the Banach-Stone Theorem (see Theorem 1.2.10). If $T : C(\Omega) \longrightarrow C(\Omega)$ is a surjective isometry, then there exist a homeomorphism $\phi : \Omega \longrightarrow \Omega$ and a continuous map $u : \Omega \longrightarrow \mathbb{T}$ such that

$$Tf(\omega) = u(\omega)f(\phi(\omega)), \quad \forall\ f \in C_0(\Omega),\ \omega \in \Omega.$$

Let P_0 be a generalized 3-circular projection on X. Then as in Definition 1.1.2, we will refer to T and λ_1, λ_2, the isometry and λ_1, λ_2 associated with the generalized 3-circular projection P_0 respectively.

We also show that if P_0 is a generalized 3-circular projection on $C(\Omega)$, then λ_1, λ_2 associated with P_0 are cube roots of unity.

All the results of this chapter appeared in [2].

Chapter 4 gives complete description of generalized 3-circular projections on $\mathbb{C}^n$ with a symmetric norm and on spaces of matrices with a unitarily invariant norm and unitary congruence invariant norm.

The descriptions of GBPs on the above spaces with the said norms are given in [16].

A norm $\|\cdot\|$ on $\mathbb{C}^n$ is called symmetric if $\|\Pi x\| = \|x\|$ for all $x \in \mathbb{C}^n$ and all permutation matrices Π. A norm $\|\cdot\|$ on $\mathbb{M}_{m,n}(\mathbb{C})$ is called unitarily invariant if $\|UAV\| = \|A\|$, for all $A \in \mathbb{M}_{m,n}(\mathbb{C})$ and all unitary matrices U and V in $\mathbb{M}_m(\mathbb{C})$ and $\mathbb{M}_n(\mathbb{C})$. A unitarily invariant norm on $\mathbb{M}_{m,n}(\mathbb{C})$ is also referred as symmetric norms (see [8]). Let $S_n(\mathbb{C})$ be the set of all $n \times n$ symmetric matrices over $\mathbb{C}$. A norm $\|\cdot\|$ on $S_n(\mathbb{C})$ is called unitary congruence invariant if $\|U^tAU\| = \|A\|$ for all $A \in S_n(\mathbb{C})$, where U is any unitary matrix in $\mathbb{M}_n(\mathbb{C})$.

If P_0 is a generalized 3-circular projection $\mathbb{C}^n$ with a symmetric norm, then we show that P_0 is either a bi-circular projection or λ_1, λ_2 associated with P_0 are cube roots of unity. We actually find the complete structure of P_0.

In case of unitarily invariant norms on $\mathbb{M}_{m,n}(\mathbb{C})$, the structure of generalized 3-circular projections depends on the isometry group and on $\lambda_1 + \lambda_2$. Let $U(X)$ denote the set of all unitary operators on a Banach space X. It is known that (see Theorem 1.2.25) if $m \neq n$, then any isometry T is of the form $T(A) = UAV$ where $U \in U(\mathbb{C}^m)$ and $V \in U(\mathbb{C}^n)$. If $m = n$, then an isometry T on $\mathbb{M}_n(\mathbb{C})$ has the form either $T(A) = UAV$ or $T(A) = UA^tV$ where U, V are unitaries in $\mathbb{M}_n(\mathbb{C})$ and A^t denotes the transpose of a matrix A.

We prove that if the isometry associated with a generalized 3-circular projection P_0 is of the form $A \longmapsto UAV$ for some $U \in U(\mathbb{C}^m)$ and $V \in U(\mathbb{C}^n)$ and $\lambda_1 + \lambda_2 = -1$, then P_0 has the form $A \longmapsto R_0AS_0 + R_1AS_1 + R_2AS_2$, where $R_i = R_i^* = R_i^2$ in $\mathbb{M}_m(\mathbb{C})$ and $S_i = S_i^* = S_i^2$ in $\mathbb{M}_n(\mathbb{C})$, $i = 0, 1, 2$.

If the isometry associated with P_0 has the same form as above and $\lambda_1 + \lambda_2 \neq -1$, then one of the following holds:

(*a*) P_0 is a bi-circular projection,

(*b*) $P_0(A) = \frac{\lambda_1 A}{2(\lambda_1 - 1)} + \frac{UAV}{1-\lambda_1^2} + \frac{\lambda_1 U^q A V^q}{2(1+\lambda_1)}$,

(*c*) $P_0(A) = \sum_{i=0}^{p-1} R_iAS_i$ for some $R_i = R_i^* = R_i^2$ in $\mathbb{M}_m(\mathbb{C})$ and $S_i = S_i^* = S_i^2$ in $\mathbb{M}_n(\mathbb{C})$, $i = 0, 1, \ldots, p-1$, and some $p > 3$.

If the isometry associated with P_0 has the form $A \longmapsto UA^tV$, then we get similar results as above.

The structures of generalized 3-circular projections for unitary congruence invariant norms will be also of similar nature.

The results for symmetric norms on $\mathbb{C}^n$ and on $\mathbb{M}_{m,n}(\mathbb{C})$ are from [3].

In Chapter 5 we discuss questions related to algebraic reflexivity of the set of GBPs and the set of isometries on spaces of Banach space valued continuous functions on a compact

Hausdorff space.

The notion of algebraic reflexivity was first introduced in [17].

Definition 1.1.3. *Let X be a Banach space and $\mathcal{S}$ a subset of $B(X)$. The algebraic closure $\overline{\mathcal{S}}^a$ of $\mathcal{S}$ is defined to be the set*

$$\{T \in B(X) : \forall\ x \in X,\ \exists\ T_x \in \mathcal{S} \text{ such that } T(x) = T_x(x)\}.$$

$\mathcal{S}$ is said to be algebraically reflexive if $\mathcal{S} = \overline{\mathcal{S}}^a$.

Algebraic reflexivity in general and on certain class of isometries were studied by many authors, see for instance [12, 13, 14, 17, 20, 23, 29, 30, 31]. Lecture Notes by Molnar [28] gives a very comprehensive account of this theory.

For a Banach space X, let $\mathcal{G}^n(X) = \{T \in \mathcal{G}(X) : T^n = I\}$. In [14], the authors proved that for a compact Hausdorff space Ω, if $\mathcal{G}(C(\Omega))$ is algebraically reflexive, then $\mathcal{G}^2(C(\Omega))$ is also algebraically reflexive. We prove this result for vector valued continuous functions and for any $n \geq 2$.

The algebraic reflexivity of the set of generalized 3-circular projections on $C(\Omega, X)$ is still open.

Remark 1.1.4. The techniques used to describe generalized 3-circular projections in Chapter 3 and Chapter 4 can be applied to describe generalized n-circular projections as well, $n > 3$. However, it is evident from the proofs that the number of cases occurring becomes increasingly large and difficult to handle. It seems that one needs some other approach to deal with such problems for general n.

1.2 Notation and Preliminaries

In this section, we introduce some notation and recall some definitions and results that will be used throughout this thesis.

Throughout this thesis we will assume X to be a complex Banach space. We will denote by $\mathbb{T}$, the unit circle in the complex plane.

We begin by recalling Definition 1.1.2.

Definition 1.2.1. A projection P_0 on X is said to be a generalized n-circular projection, $n \geq 2$, if there exist $\lambda_1, \lambda_2, \ldots, \lambda_{n-1} \in \mathbb{T} \setminus \{1\}$, λ_i, $i = 1, 2, \ldots, n-1$ which are of finite order and projections $P_0, P_1, \ldots, P_{n-1}$ on X such that

(a) $\lambda_i \neq \lambda_j$ for $i \neq j$,

(b) $P_0 \oplus P_1 \oplus \cdots \oplus P_{n-1} = I$,

(c) $P_0 + \lambda_1 P_1 + \cdots + \lambda_{n-1} P_{n-1}$ is a surjective isometry.

Let P_0 be a generalized n-circular projection, that is, $P_0 + \lambda_1 P_1 + \cdots + \lambda_{n-1} P_{n-1} = T$ for some surjective isometry T and λ_i, P_i are as in Definition 1.2.1, $i = 1, 2, \ldots, n-1$.

Suppose that $\lambda_m = \lambda_n$ for some m, n then we see that

$$T = P_0 + \lambda_1 P_1 + \cdots + \lambda_m (P_m + P_n) + \cdots + \lambda_{n-1} P_{n-1}.$$

As $(P_m + P_n)$ is a projection, we conclude that P_0 is a generalized $(n-1)$-circular projection.

Similarly, if $\lambda_m = -\lambda_n$, then $(P_m - P_n)$ is not a projection but $(P_m - P_n)^2 = P_m + P_n$ is. Therefore, we have

$$T = P_0 + \lambda_1 P_1 + \cdots + \lambda_m (P_m - P_n) + \cdots + \lambda_{n-1} P_{n-1}.$$

This implies that

$$T^2 = P_0 + \lambda_1^2 P_1 + \cdots + \lambda_m^2 (P_m + P_n) + \cdots + \lambda_{n-1}^2 P_{n-1}.$$

Since T^2 is an isometry, we conclude that P_0 is a generalized $(n-1)$-circular projection. So, we define the following.

Definition 1.2.2. *A generalized n-circular projection P_0 is called proper if it is not a generalized $(n-1)$-circular projection.*

Botelho in [9] introduced the notion of generalized n-circular projection as follows:

A projection P on X is said to be a generalized n-circular projection if there exists a surjective isometry T of order n such that

$$P = \frac{I + T + \cdots + T^{n-1}}{n}.$$

Remark 1.2.3. *Let $T \in \mathcal{G}(X)$ such that $T^n = I$ and $P = \frac{I+T+\cdots+T^{n-1}}{n}$. Then P is a generalized n-circular projection in the sense of Definition 1.2.1.*

To see this, we first note that P is a projection. Let $\lambda_0 = 1, \lambda_1, \ldots, \lambda_{n-1}$ be the n distinct roots of identity. For $i = 1, 2, \ldots, n-1$, we define

$$P_i = \frac{I + \overline{\lambda_i}T + \cdots + \overline{\lambda_i}^{n-1}T^{n-1}}{n}.$$

Then each P_i is a projection, $P_0 \oplus P_1 \oplus \cdots \oplus P_{n-1} = I$ and $P_0 + \lambda_1 P_1 + \cdots + \lambda_{n-1}P_{n-1} = T$.

Definition 1.2.4. *A projection P on a Banach space X is said to be bi-circular projection if $P + \lambda(I - P)$ is a surjective isometry, for all $\lambda \in \mathbb{T}$.*

Definition 1.2.5. *A projection on a Banach space X is said to be generalized bi-circular projection if there exists a $\lambda \in \mathbb{T} \setminus \{1\}$ such that $P + \lambda(I - P)$ is a surjective isometry.*

Generalized bi-circular projections are not necessarily Hermitian. If P is a GBP, then so is $I - P$.

Remark 1.2.6. *We note that in Definition 1.2.5, it is not necessary to assume that $P + \lambda(I - P)$ is surjective. It follows that this isometry is always surjective. To see this, let $x \in X$ and $y = Px + \frac{1}{\lambda}(I - P)x$. Then we have $(P + \lambda(I - P))(y) = Px + (I - P)x = x$.*

Theorem 1.2.7. *[26, Theorem 1] Let X be a complex Banach space and P a projection on X. Suppose that $P + \lambda(I - P)$ is an isometry. If λ is of infinite order in $\mathbb{T}$, then P is Hermitian.*

Definition 1.2.8. *Let P be a generalized bi-circular projection on X. The multiplicative group associated with P is defined to be the set*

$$\Lambda_P = \{\lambda \in \mathbb{T} : P + \lambda(I - P) \text{ is an isometry}\}.$$

This set is a group under multiplication.

The relation between finite order operators and projections is given in the next theorem.

Theorem 1.2.9. *Let X be a Banach space and T an operator of order n. Then there exist pairwise orthogonal projections P_i, $i = 0, 1, \ldots, n-1$ such that $T = P_0 + \lambda_1 P_1 + \cdots + \lambda_{n-1} P_{n-1}$; where $\lambda_1, \ldots, \lambda_{n-1}$ are $(n-1)$ roots of unity.*

The proof follows from [36, Theorem 5.9-E].

We denote by $C(\Omega, X)$, the space of X-valued continuous functions on compact Hausdorff space Ω while $C_0(\Omega, X)$ denotes the space of X-valued continuous functions on a locally compact Hausdorff space Ω which vanish at infinity. Both $C(\Omega, X)$ and $C_0(\Omega, X)$ are equipped with the supremum norm, that is, $\|f\|_\infty = \sup_{\omega \in \Omega} \|f(\omega)\|$. If $X = \mathbb{C}$, the above spaces are denoted by $C(\Omega)$ and $C_0(\Omega)$ respectively.

We denote by $U(X)$ and $\mathcal{G}(X)$ the group of unitary operators and surjective linear isometries on X respectively.

Theorem 1.2.10. *[4, Theorem 7.1] Let Ω be a locally compact Hausdorff space. If $T : C_0(\Omega) \longrightarrow C_0(\Omega)$ is a surjective isometry, then there exist a homeomorphism $\phi : \Omega \longrightarrow \Omega$ and a continuous map $u : \Omega \longrightarrow \mathbb{T}$ such that*

$$Tf(\omega) = u(\omega) f(\phi(\omega)), \quad \forall\ f \in C_0(\Omega),\ \omega \in \Omega.$$

For the vector-valued version of the above theorem we recall the notion of a centralizer of a Banach space, see [18, Chapter I].

Definition 1.2.11. *Let T be a bounded linear operator on a Banach space X.*

(*i*) *The operator T is called a multiplier of X if for every element $p \in ext(B_{X^*})$, there exists $a_T(p) \in \mathbb{C}$ such that $T^* p = a_T(p) p$. The collection of all multipliers is denoted by $Mult(X)$.*

(*ii*) *The centralizer of X is defined as*

$$Z(X) = \{T \in Mult(X) : \exists\ \overline{T} \in Mult(X) \text{ such that } a_{\overline{T}}(p) = \overline{a_T(p)},\ \forall\ p \in ext(B_{X^*})\}.$$

Definition 1.2.12. *A Banach space X is said to have trivial centralizer if the dimension of $Z(X)$ is equal to 1; that is, if the only elements in the centralizer are scalar multiples of the identity operator I. Obviously, this is true if X is itself the scalar field.*

Theorem 1.2.13. *[4, Theorem 8.10] Let Ω be a locally compact Hausdorff space and X a Banach with trivial centralizer. If $T : C_0(\Omega, X) \longrightarrow C_0(\Omega, X)$ is a surjective isometry, then there exist a homeomorphism $\phi : \Omega \longrightarrow \Omega$ and a map $u : \Omega \longrightarrow \mathcal{G}(X)$, continuous with respect to the strong operator topology of $B(X)$, such that*

$$Tf(\omega) = u_\omega(f(\phi(\omega))), \quad \forall\ f \in C_0(\Omega),\ \omega \in \Omega.$$

For simplicity, we denote $u(\omega)$ by u_ω.

Definition 1.2.14. *[4, Definition 8.2] A Banach space X is said to have the strong Banach-Stone property if it satisfies the condition in Theorem 1.2.13.*

It is known that strictly convex spaces have trivial centralizer. In particular, they have the strong Banach-Stone property.

The following theorem describes GBPs on $C(\Omega, X)$.

Theorem 1.2.15. *[11, Theorem 2.1] If Ω is a connected compact Hausdorff space and X has the strong Banach-Stone property, then Q is a generalized bi-circular projection on $C(\Omega, X)$ if and only if one of the following statements holds:*

1. *There exist a nontrivial homeomorphism $\phi : \Omega \longrightarrow \Omega$ with $\phi^2 = Id$ and a continuous function $u : \Omega \longrightarrow \mathcal{G}(X)$ with $u_\omega \circ u_{\phi(\omega)} = Id$ such that*

$$Q(f)(\omega) = \frac{1}{2}[f(\omega) + u_\omega(f \circ \phi(\omega))],$$

for every $\omega \in \Omega$.

2. *There exists a generalized bi-circular projection on X, P_ω, such that $Q(f)(\omega) = P_\omega(f(\omega))$, for each $\omega \in \Omega$.*

Definition 1.2.16. *A projection P on a Banach space X is said to be an L_∞ projection if for every $x \in X$*

$$\|x\| = \max\{\|Px\|, \|x - Px\|\}.$$

X has trivial L_∞-structure if 0 and I are the only L_∞ projections.

Theorem 1.2.17. *[15, Theorem 2.5] Let (X_n) be a sequence of complex Banach spaces such that every X_n has trivial L_∞-structure. T is a surjective isometry of $\bigoplus_{c_0} X_n$ if and only if there exist a permutation π of $\mathbb{N}$ and a sequence of isometric operators $U_{n\pi(n)}$ such that*

$$(Tx)_n = U_{n\pi(n)} x_{\pi(n)} \text{ for each } x = (x_n) \in \bigoplus_{c_0} X_n.$$

Moreover, the space $X_{\pi(n)} \cong X_n$.

We now recall some definitions and remarks which will be used in Chapter 4.

Definition 1.2.18. *Let X be a Banach space and G a closed subgroup of $\mathcal{G}(X)$. A norm $\|\cdot\|$ on X is said to be G-invariant if*

$$\|g(x)\| \;=\; \|x\| \quad \forall\, g \in G,\; x \in X.$$

Trivially, multiple of the inner product norm on $\mathbb{C}^n$ is G-invariant.

The following theorem shows that GBPs on finite dimensional Banach spaces are orthogonal projections.

Theorem 1.2.19. *[16, Proposition 2.1] Let X be an n-dimensional inner product space and $\|\cdot\|$ a multiple of the norm induced by the inner product. Suppose $P : X \longrightarrow X$ is a non-trivial linear projection and $\lambda \in \mathbb{T} \setminus \{1\}$. The following conditions are equivalent:*

(i) $P + \lambda(I - P)$ is an isometry,

(ii) P is an orthogonal projection, that is, there exists an orthonormal basis $\{e_1, \ldots, e_n\}$ for X such that $P(e_j) = \lambda_j e_j$ where $\lambda_j \in \{0, 1\}$ for all $j = 1, \ldots, n$.

Remark 1.2.20. In the sequel, we will prove our results for G-invariant norms which are not multiple of the inner product norm.

Definition 1.2.21. *A square matrix P is called a permutation matrix if exactly one entry in each row and column is equal to 1, and all other entries are 0.*

Every permutation matrix corresponds to a unique permutation. A permutation matrix will always be in the form

$$\begin{bmatrix} e_{a_1} \\ e_{a_2} \\ \vdots \\ e_{a_n} \end{bmatrix}$$

where e_{a_j} denotes a row vector of length n with 1 in the j^{th} position and 0 in every other position and

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ a_1 & a_1 & \cdots & a_n \end{pmatrix}$$

is the corresponding permutation form of the permutation matrix.

Definition 1.2.22. *A norm $\|\cdot\|$ on $\mathbb{C}^n$ is called symmetric if $\|\Pi x\| = \|x\|$ for all $x \in \mathbb{C}^n$ and all permutation matrices Π.*

Let G be the group of all generalized permutation matrices, that is, matrices of the form DP where D is a diagonal matrix with all its elements of unit modulus and P is a permutation matrix.

The isometry group of a given symmetric norm is characterized in the following theorem.

Theorem 1.2.23. *[24, Theorem 2.5] The isometry group of a symmetric norm on $\mathbb{C}^n$ is G.*

Definition 1.2.24. *A norm $\|\cdot\|$ on $\mathbb{M}_{m,n}(\mathbb{C})$ is called unitarily invariant if $\|UAV\| = \|A\|$, for all $A \in \mathbb{M}_{m,n}(\mathbb{C})$ and all unitary matrices U and V in $\mathbb{M}_m(\mathbb{C})$ and $\mathbb{M}_n(\mathbb{C})$ respectively.*

Let G be the group of all linear operators on $\mathbb{M}_{m,n}(\mathbb{C})$ of the form $A \longmapsto UAV$ for some fixed unitary $U \in \mathbb{M}_m(\mathbb{C})$ and $V \in \mathbb{M}_n(\mathbb{C})$.

We denote by τ the transposition operator on $\mathbb{M}_n(\mathbb{C})$, that is, $\tau(A) = A^t$.

The isometry group of a unitarily invariant norm is described in the following theorem by Li.

Theorem 1.2.25. *[24, Theorem 2.4] The isometry group of a unitarily invariant norm $\|\cdot\|$ on $\mathbb{M}_{m,n}(\mathbb{C})$ must be of one of the following forms:*

(a) If $m \neq n$, $\mathcal{G}(X) = G$;

(b) If $m = n$, $\mathcal{G}(X) = \langle G, \tau \rangle$.

The following proposition will be used in Chapter 4.

Proposition 1.2.26. *[16, Proposition 4.1] Let $\|\cdot\|$ be a unitarily invariant norm on $\mathbb{M}_{m,n}(\mathbb{C})$ not equal to a multiple of the Frobenius norm, and $\mathcal{K}$ the isometry group of $\|\cdot\|$. Suppose $P : \mathbb{M}_{m,n}(\mathbb{C}) \longrightarrow \mathbb{M}_{m,n}(\mathbb{C})$ is a non-trivial linear projection and $\lambda \in \mathbb{T} \setminus \{1\}$. Then $P + \lambda(I - P) \in \mathcal{K}$ if and only if one of the following holds:*

(a) There exists $R \in \mathbb{M}_m(\mathbb{C})$ with $R = R^* = R^2$ such that P has the form $A \longmapsto RA$ or there exists $S \in \mathbb{M}_n(\mathbb{C})$ with $S = S^* = S^2$ such that P has the form $A \longmapsto AS$.

(b) $\lambda = -1$, and there exist $R = R^* = R^2$ in $\mathbb{M}_m(\mathbb{C})$ and $S = S^* = S^2$ in $\mathbb{M}_n(\mathbb{C})$ such that P has the form $A \longmapsto RAS + (I_m - R)A(I_n - S)$.

(c) $m = n$, $\lambda = -1$, and there is $U \in U(\mathbb{C}^n)$ such that P or $\overline{P}$ has the form $A \longmapsto (A + UA^t\overline{A})/2$.

Definition 1.2.27. *A norm $\|\cdot\|$ on $S_n(\mathbb{C})$, the space of all $n \times n$ symmetric matrices over $\mathbb{C}$, is called unitary congruence invariant if $\|U^tAU\| = \|A\|$ for all $A \in S_n(\mathbb{C})$, where U is an any unitary matrix in $\mathbb{M}_n(\mathbb{C})$.*

Let G be the group of all linear operators on $S_n(\mathbb{C})$ of the form $A \longmapsto U^tAU$ for some fixed unitary $U \in \mathbb{M}_n(\mathbb{C})$.

The isometry group of a unitary congruence invariant norm on $S_n(\mathbb{C})$ is described in the following theorem.

Theorem 1.2.28. *[24, Theorem 2.8] The isometry group of a unitary congruence invariant norm on $S_n(\mathbb{C})$ which is not a multiple of the Frobenius norm is G.*

The following proposition will be used in Chapter 4.

Proposition 1.2.29. *[16, Proposition 5.1] Let $\|\cdot\|$ be a unitary congruence invariant norm on $S_n(\mathbb{C})$, which is not a multiple of the Frobenius norm, and $\mathcal{K}$ the isometry group of $\|\cdot\|$. Suppose $P : S_n(\mathbb{C}) \longrightarrow S_n(\mathbb{C})$ is a non-trivial linear projection and $\lambda \in \mathbb{T} \setminus \{1\}$. Then $P + \lambda(I - P) \in \mathcal{K}$ if and only if $\lambda = -1$ and there exists $R = R^* = R^2$ in $\mathbb{M}_n(\mathbb{C})$ such that P or $\overline{P}$ has the form $A \longmapsto R^t AR + (I - R^t)A(I - R)$.*

The next theorem gives sufficient condition regarding the algebraic reflexivity of the set of isometric reflections on $C(\Omega)$.

Theorem 1.2.30. *[14, Theorem 1] Let Ω be compact Hausdorff space. If $\mathcal{G}(C(\Omega))$ is algebraically reflexive, then $\mathcal{G}^2(C(\Omega))$ is also algebraically reflexive.*

The following theorem gives conditions on X so that $\mathcal{G}(C(\Omega, X))$ is algebraically reflexive.

Theorem 1.2.31. *[20, Theorem 7] Suppose Ω is a first countable compact Hausdorff space and X a uniformly convex Banach space such that $\mathcal{G}(X)$ is algebraically reflexive . Then $\mathcal{G}(C(\Omega, X))$ is algebraically reflexive.*

CHAPTER 2

Representation of Generalized Bi-circular Projections

In this chapter we prove several results concerning the representation of projections on Banach spaces. We characterize projections written as combination of powers of a finite order operator and relate those to generalized n-circular projections. We also characterize generalized bi-circular projections on $C_0(\Omega, X)$, with Ω not necessarily connected and X a Banach space with trivial centralizer.

All the results of this chapter had appeared in [1].

2.1 A characterization of generalized bi-circular projection

We recall from Definition 1.2.5 that a projection P on a Banach space X is said to be a generalized bi-circular projection if there exists a $\lambda \in \mathbb{T} \setminus \{1\}$ such that $P + \lambda(I - P)$ is an isometry on X.

We start with the following remark.

Remark 2.1.1. *Let P and R be operators on X such that $P = \frac{I+R}{2}$. Then P is a projection if and only if R is a reflection.*

So, there is a bijection between the set of all reflections on X and the set of all projections. If $P = \frac{I+R}{2}$, with R a reflection, then R is the identity on the range of P and $-I$ on the kernel of P.

Given a projection P on X, $2P - I$ is a reflection, and thus P can be represented as the average of I with a reflection, that is, $P = \frac{I+(2P-I)}{2}$. In particular, any generalized bi-circular projection on X is average of the identity operator and a reflection. In the next result we show that, if P is a GBP and T is the isometry associated with P, then $2P - I$ belongs to the algebra generated by T.

Theorem 2.1.2. *Let X be a Banach space. If P is a projection such that $P+\lambda(I-P) = T$, where $\lambda \in \mathbb{T} \setminus \{1\}$ is of finite order and T is an isometry on X, then $R = 2P - I$ belongs to the algebra generated by T.*

Proof. We consider the following two cases:

(a) Suppose that λ is of even order. Then for some positive integer k, we have $\lambda^k = -1$, $P + \lambda^k(I - P) = T^k$ and $P + \lambda^{2k}(I - P) = I = T^{2k}$. Consequently, P is represented as the average of the identity with the isometric reflection T^k.

(b) Suppose that the order of λ is $2k + 1$, $k \geq 1$. Therefore, we have

$$P + \lambda^j(I - P) = T^j, \quad \forall\ j = 1, \ldots, 2k + 1. \tag{2.1.1}$$

From Equation (2.1.1) we get

$$(2k + 1)P + (1 + \lambda + \lambda^2 + \cdots + \lambda^{2k})(I - P) = I + T + T^2 + \cdots + T^{2k}.$$

Since $1 + \lambda + \lambda^2 + \cdots + \lambda^{2k} = 0$, we obtain

$$(2k + 1)P = I + T + T^2 + \cdots + T^{2k}. \tag{2.1.2}$$

Equation (2.1.2) implies that

$$P = \frac{1}{2k+1}\left(I + T + \cdots + T^{2k}\right) = \frac{I+R}{2},$$

with

$$R = \frac{(1-2k)I + 2T + \cdots + 2T^{2k}}{2k+1} = 2P - I.$$

This completes the proof. $\square$

Corollary 2.1.3. *Let X be a Banach space. If the order of the multiplicative group of a generalized bi-circular projection P on X is even, then P is the average of the identity with an isometric reflection.*

Remark 2.1.4. If P is a projection such that $P + \lambda(I - P) = T$, where $\lambda \in \mathbb{T} \setminus \{1\}$ is of infinite order, then it follows from Theorem 1.2.7 that P is a bi-circular projection.

Remark 2.1.5. *It follows from proof of the above Theorem that every GBP is a generalized n-circular projection, where n is the order of the λ associated with the GBP.*

We now give an example which shows that, for a projection P, if $2P - I$ belongs to the algebra generated by an isometry, then P need not be a GBP.

Example 2.1.6. *Let X be the space of all convergent sequences in $\mathbb{C}$ with the sup norm. Let $T : X \longrightarrow X$ be given by*

$$T(x_1, x_2, x_3, x_4, \ldots) = (x_2, x_3, x_1, x_4, \ldots),$$

which involves a permutation of the first three positions of a sequence in X and the identity at any other position. Let $P = \frac{I+T+T^2}{3}$. It is clear that T is a surjective isometry and

$$P(x_1, x_2, x_3, x_4, \ldots) = \left(\frac{x_1+x_2+x_3}{3}, \frac{x_1+x_2+x_3}{3}, \frac{x_1+x_2+x_3}{3}, x_4, \ldots\right)$$

is a projection. We set $R = 2P - I$. This implies that

$$R(x_1, x_2, x_3, x_4, \ldots) = \left(\frac{-x_1+2x_2+2x_3}{3}, \frac{2x_1-x_2+2x_3}{3}, \frac{2x_1+2x_2-x_3}{3}, x_4, \ldots\right).$$

Therefore, we have $R(0,1,1,0,\ldots) = \frac{1}{3}(4,1,1,0,0,\ldots)$, *which shows that* R *is not an isometry.*

We claim that P *is not a GBP. To see this, given* λ *of modulus* 1 *and* $\lambda \neq 1$, *we set* $S = P + \lambda(I - P)$. *In particular, we have*

$$S(1,0,0,0,\ldots) = \left(\frac{1}{3} + \frac{2}{3}\lambda, \frac{1}{3} - \frac{1}{3}\lambda, \frac{1}{3} - \frac{1}{3}\lambda, 0, \ldots\right).$$

If S *is an isometry on* X, *then* $\max\{|\frac{1}{3} + \frac{2}{3}\lambda|, |\frac{1}{3} - \frac{1}{3}\lambda|\} = 1$. *We observe that* $|\frac{1}{3} - \frac{1}{3}\lambda| < 1$ *and if* $|\frac{1}{3} + \frac{2}{3}\lambda| = 1$, *then* $\lambda = 1$. *This contradiction shows that* P *is not a GBP.*

Remark 2.1.7. *The projection* P *defined above is an example of a generalized* 3*-circular projection which is not a GBP. Hence, the converse of Remark 2.1.5 is not always true.*

Our next result characterizes projections which are GBPs.

Proposition 2.1.8. *Let* X *be a Banach space. Let* P *be a projection on* X *such that* $T = P + \lambda(I - P)$, *for some* $\lambda \in \mathbb{T} \setminus \{1\}$. *Then* T *is an isometry if and only if* $\|x - y\| = \|x - \lambda y\|$ *for every* $x \in$ *Range*(P) *and* $y \in$ *Ker*(P).

Proof. The projection P determines two closed subspaces, Range(P) and Ker(P) such that $X = \text{Range}(P) \oplus \text{Ker}(P)$. Since T is an isometry, $\|x - y\| = \|Tx - Ty\|$ for every x and y in X. In particular, for x in the range of P and y in the kernel of P, we have $Tx = x$ and $Ty = \lambda y$.

Conversely, for every $x \in \text{Range}(P)$ and $y \in \text{Ker}(P)$, we have $Tx = x$ and $Ty = \lambda y$. Therefore, $\|x - y\| = \|x - \lambda y\| = \|Tx - Ty\|$. □

Corollary 2.1.9. *A generalized bi-circular projection* P *on* X *is the average of the identity with an isometric reflection if and only if for every* $x \in$ *Range*(P) *and* $y \in$ *Ker*(P), $\|x - y\| = \|x + y\|$.

The next proposition asserts that every projection on a Banach space is a generalized bi-circular projection in some equivalent renorming of the given space.

Proposition 2.1.10. *Let P be a projection on a Banach space X. Then X can be equivalently renormed so that $2P-I$ is an isometric reflection and consequently, P is a generalized bi-circular projection.*

Proof. We set $R = 2P-I$ and observe that $R^2 = I$. This implies that R is an isomorphism. We define $\|x\|_1 = \|x\| + \|R(x)\|$, for all $x \in X$. This new norm is equivalent to the original norm on X and R relative to this norm is an isometry. In fact we have that, given $x \in X$, $\|R(x)\|_1 = \|R(x)\| + \|R(R(x))\| = \|x\|_1$. □

We now give an example to show that the λ associated with a GBP may not be always -1.

Example 2.1.11. *Let X be $\mathbb{C}^3$ with the max norm, $\|(x,y,z)\|_\infty = \max\{|x|,|y|,|z|\}$ and $\lambda = exp(\frac{2\pi i}{3}) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$.*

We consider the following projection P on $\mathbb{C}^3$:

$$P(x,y,z) = \frac{1}{3}(x+y+z, x+y+z, x+y+z).$$

Let $T = P + \lambda(I-P)$. Straightforward computations imply that

$$T(x,y,z) = (a\,x + b\,(y+z), a\,y + b\,(x+z), a\,z + b\,(x+y))\,,$$

with $a = \frac{i\sqrt{3}}{3}$ and $b = \frac{1}{2} - \frac{\sqrt{3}i}{6}$.

Since we have $T(0,0,1) = (b,b,a)$, T is not an isometry. In fact, $\|(0,0,1)\|_\infty = 1$ and $\|T(0,0,1)\|_\infty = \frac{\sqrt{3}}{3} \neq \|(0,0,1)\|_\infty$. The operator T has order 3 since $\lambda^3 = 1$.

We now renorm $\mathbb{C}^3$ so that T becomes an isometry. The new norm is defined as follows:

$$\|(x,y,z)\|_1 = \max\{\|(x,y,z)\|_\infty,\ \|T(x,y,z)\|_\infty, \|T^2(x,y,z)\|_\infty\}.$$

This implies that

$$\|T(x,y,z)\|_1 = \max\{\|T(x,y,z)\|_\infty,\ \|T^2(x,y,z)\|_\infty, \|T^3(x,y,z)\|_\infty\}.$$

But we have $T^3 = I$, so we get

$$\|T(x,y,z)\|_1 = \|(x,y,z)\|_1.$$

Hence, T is an isometry with the norm $\|\cdot\|_1$ and therefore, P is a GBP in $\mathbb{C}^3$. We claim that P can not be written as the average of the identity with an isometric reflection. To see this, assume on the contrary that $P = \frac{I+R}{2}$. Hence, we have $R = 2P - I$. We show that R is not an isometry. We observe that

$$\begin{aligned} R(0,0,1) &= (2/3, 2/3, -1/3), \\ (TR)(0,0,1) &= \frac{1}{3}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i, \frac{1}{2} + \frac{\sqrt{3}}{2}i, 2 - i\sqrt{3}\right) \text{ and} \\ (T^2R)(0,0,1) &= \frac{1}{3}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i, \frac{1}{2} - \frac{\sqrt{3}}{2}i, 2 + i\sqrt{3}\right). \end{aligned}$$

Since we have

$$\|R(0,0,1)\|_\infty = \frac{2}{3}, \ \|(TR)(0,0,1)\|_\infty = \|(T^2R)(0,0,1)\|_\infty = \frac{\sqrt{7}}{3},$$

we conclude that

$$\|R(0,0,1)\|_1 = \max\left\{\frac{2}{3}, \frac{\sqrt{7}}{3}\right\} = \frac{\sqrt{7}}{3} \neq \|(0,0,1)\|_1 = 1.$$

Remark 2.1.12. *It is worth mentioning that the projection P above does not satisfy the condition stated in Corollary 2.1.9. For example, if $x = (1,1,1) \in Range(P)$, $y = (1,1,-2) \in Ker(P)$, we have $\|x+y\|_1 = \sqrt{7}$ and $\|x-y\|_1 = 3$.*

2.2 Projections as combination of finite order operators

In this section we investigate the existence of projection defined as linear combination of the powers of a given finite order operator. We conclude in Theorem 2.2.2 that only certain averages yield projections.

We recall that the multiplicative group associated with a GBP P is defined to be the set

$$\Lambda_P = \{\lambda \in \mathbb{T} : P + \lambda(I - P) \text{ is an isometry}\}.$$

From the proof of Theorem 2.1.2, it follows that Λ_P is either finite or equal to $\mathbb{T}$. If Λ_P is infinite, then P is a bi-circular projection. We give some examples of GBPs together with their multiplicative groups.

Example 2.2.1. *1. Consider ℓ_∞ with the usual sup norm. Let P be defined as follows:*

$$P(x_1, x_2, x_3, \ldots) = \left(\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2}, x_3, \cdots \right).$$

We show that $\Lambda_P = \{1, -1\}$. Given $\lambda \in \mathbb{T}$ such that $T = P + \lambda(I - P)$ is a surjective isometry, then $T(x_1, x_2, x_3, \ldots)$

$$= \left(\frac{(1+\lambda)x_1 + (1-\lambda)x_2}{2}, \frac{(1+\lambda)x_2 + (1-\lambda)x_1}{2}, x_3, \cdots \right).$$

Theorem 1.2.10 implies that a surjective isometry S on ℓ_∞ is of the form

$$S(x_1, x_2, x_3, \ldots) = (\mu_1 x_{\tau(1)}, \mu_2 x_{\tau(2)}, \ldots),$$

with τ a bijection of $\mathbb{N}$ and $\{\mu_i\}$ is a sequence of modulus 1 complex numbers. It follows that T is an isometry if and only if $1 + \lambda = 0$ or $1 - \lambda = 0$. Hence, we have $\lambda = \pm 1$.

2. Let P and T on $(\mathbb{C}^3, \|\cdot\|_1)$ be defined as in Example 2.1.11. Then we have $\Lambda_P = \{1, exp(\frac{2\pi i}{3}), exp(\frac{4\pi i}{3})\}$. Since we have $T = P + exp(\frac{2\pi i}{3})(I - P)$ is an isometry on $(\mathbb{C}^3, \|\cdot\|_1)$, it follows that $T^2 = P + exp(\frac{4\pi i}{3})(I - P)$ is also an isometry and $\Lambda_P \supseteq \{1, exp(\frac{2\pi i}{3}), exp(\frac{4\pi i}{3})\}$. We now show that $\Lambda_P = \{1, exp(\frac{2\pi i}{3}), exp(\frac{4\pi i}{3})\}$. As in Example 2.1.11, let $\lambda = exp(\frac{2\pi i}{3})$. Given $\lambda_0 = a_0 + ib_0$ of modulus 1 such that $\lambda_0 \notin \{1, exp(\frac{2\pi i}{3}), exp(\frac{4\pi i}{3})\}$, we set $S = P + \lambda_0(I - P)$. Therefore, we have

$$S(x, y, z) = \frac{1}{3}(cx + d(y + z), cy + d(x + z), cz + d(x + y)),$$

with $c = 1 + 2\lambda_0$ and $d = 1 - \lambda_0$ and

$$\|S(0,0,1)\|_1 = \max\{\|S(0,0,1)\|_\infty, \|TS(0,0,1)\|_\infty, \|T^2S(0,0,1)\|_\infty\}.$$

We now have

$$S(0,0,1) = \frac{1}{3}(d, d, c),\ TS(0,0,1) = \frac{1}{3}(1 - \lambda_0\lambda, 1 - \lambda_0\lambda, 1 + 2\lambda_0\lambda)$$

and

$$T^2S(0,0,1) = \frac{1}{3}(1 - \lambda_0\lambda^2, 1 - \lambda_0\lambda^2, 1 + 2\lambda_0\lambda^2).$$

It is easy to see that each of $|\frac{1-\lambda_0}{3}|$, $|\frac{1-\lambda_0\lambda}{3}|$ *and* $|\frac{1-\lambda_0\lambda^2}{3}|$ *is strictly less than 1. Moreover, if any of* $|\frac{1+2\lambda_0}{3}|$, $|\frac{1+2\lambda_0\lambda}{3}|$ *or* $|\frac{1+2\lambda_0\lambda^2}{3}|$ *is equal to 1, then* $\lambda_0 = 1$, $\lambda_0 = \overline{\lambda}$ *or* $\lambda_0 = \overline{\lambda}^2$*, respectively. This leads to a contradiction. It also follows from calculations already done in Example 2.1.11 that* $\|(0,0,1)\|_1 = 1$*. Therefore, we get* $\|S(0,0,1)\|_1 \neq \|(0,0,1)\|_1$ *and hence we conclude that* $\lambda_0 \notin \Lambda_P$*.*

We now show the main result of this section on the existence of projections written as a linear combination of operators with a cyclic property.

We recall from Introduction that, the discrete Fourier coefficient of a k-tuple $z = (z(0), \ldots, z(k-1))$ is defined as $\hat{z}(m) = \sum_{n=0}^{k-1} z(n)\rho^{mn}$, where $\rho = e^{-2\pi i/k}$. Then z is the inverse discrete Fourier transform (IDFT for short) of $\hat{z}$.

Theorem 2.2.2. *Let P a bounded operator on a Banach space X. Let $\lambda_0, \ldots, \lambda_{k-1}$ be nonzero complex numbers and $P = \sum_{i=0}^{k-1} \lambda_i T^i$, where T is an operator of order k. Then P is a projection if and only if $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_{k-1})$ is the IDFT of δ_S, for some $S \subseteq \{0, \ldots, k-1\}$.*

Proof. Suppose that $P = \sum_{i=0}^{k-1} \lambda_i T^i$, where T is an operator of order k. Then Theorem 1.2.9 asserts that

$$T = Q_0 + \rho Q_1 + \cdots + \rho^{k-1} Q_{k-1}$$

with $\{Q_0, \ldots, Q_{k-1}\}$ pairwise orthogonal projections. Since we have

$$T^i = Q_0 + \rho^i Q_1 + \cdots + \rho^{i(k-1)} Q_{k-1} = \sum_{j=0}^{k-1} \rho^{ij} Q_j,$$

we conclude that

$$\begin{aligned}
P &= \lambda_0 I + \lambda_1 T + \cdots + \lambda_{k-1} T^{k-1} \\
&= \lambda_0 (\sum_{j=0}^{k-1} Q_j) + \lambda_1 (\sum_{j=0}^{k-1} \rho^j Q_j) + \cdots + \lambda_{k-1} \sum_{j=0}^{k-1} \rho^{j(k-1)} Q_j \\
&= (\sum_{j=0}^{k-1} \lambda_j) Q_0 + (\sum_{j=0}^{k-1} \lambda_j \rho^j) Q_1 + \cdots + (\sum_{j=0}^{k-1} \lambda_j \rho^{j(k-1)}) Q_{k-1} \\
&= \alpha_0 Q_0 + \alpha_1 Q_1 + \cdots + \alpha_{k-1} Q_{k-1},
\end{aligned}$$

where $\alpha_i = \sum_{j=0}^{k-1} \lambda_j \rho^{ij}$. Since P is a projection, that is, $P^2 = P$ and $\{Q_0, \ldots, Q_{k-1}\}$ are pairwise orthogonal projections, we have that $\alpha_i^2 = \alpha_i$, for $i = 0, \ldots, k-1$. This implies that $(\lambda_0, \lambda_1, \ldots, \lambda_{k-1})$ is the $IDFT(\delta_S)$ for some subset S of $\{0, \ldots, k-1\}$.

Conversely, let $T = \sum_{i=0}^{k-1} \rho^i Q_i$. Then we have

$$\delta_S(0)Q_0 + \delta_S(1)Q_1 + \cdots + \delta_S(k-1)Q_{k-1} = \sum_{i=0}^{k-1} \lambda_i T^i$$

and

$$P = \delta_S(0)Q_0 + \delta_S(1)Q_1 + \cdots + \delta_S(k-1)Q_{k-1}.$$

This implies that $P^2 = P$ and the proof is complete. □

2.3 Spaces of vector-valued functions

In this section we characterize generalized bi-circular projections on spaces of continuous functions defined on a locally compact Hausdorff space. This characterization extends Theorem 1.2.15 to to more general settings.

Lemma 2.3.1. *Let X be a Banach space and $\lambda \in \mathbb{T} \setminus \{1\}$. Then the following assertions are equivalent:*

(*a*) *T is a bounded operator on X satisfying $T^2 - (\lambda+1)T + \lambda I = 0$.*

(*b*) *There exists a projection P on X such that $P + \lambda(I - P) = T$.*

Proof. $(a) \Longrightarrow (b)$ We define $P = \dfrac{T - \lambda I}{1 - \lambda}$. Then we have $P + \lambda(I - P) = T$. Moreover, we see that

$$\begin{aligned} P^2 &= \frac{T^2 + \lambda^2 - 2\lambda T}{(1-\lambda)^2} \\ &= \frac{(\lambda+1)T - \lambda I + \lambda^2 - 2\lambda T}{(1-\lambda)^2} \\ &= \frac{(1-\lambda)(T - \lambda I)}{(1-\lambda)^2} \\ &= P. \end{aligned}$$

$(b) \Longrightarrow (a)$

$$\begin{aligned} T^2 - (\lambda+1)T + \lambda I &= P + \lambda^2(I-P) - (\lambda+1)[P + \lambda(I-P)] + \lambda I \\ &= -\lambda P + [\lambda^2 - \lambda(\lambda+1)](I-P) + \lambda I \\ &= 0. \end{aligned}$$ □

Theorem 2.3.2. *Let Ω be a locally compact Hausdorff space and X a Banach space with trivial centralizer. Let P be a generalized bi-circular projection on $C_0(\Omega, X)$. Then one and only one of the following assertions holds:*

(a) $P = \frac{I+T}{2}$, where T is an isometry on $C_0(\Omega, X)$.

(b) $Pf(\omega) = P_\omega(f(\omega))$, where P_ω is a generalized bi-circular projection on X.

Proof. Let $P + \lambda(I-P) = T$, where $\lambda \in \mathbb{T} \setminus \{1\}$ and T is an isometry on $C_0(\Omega, X)$. From Theorem 1.2.13, T has the form

$$Tf(\omega) = u_\omega(f(\phi(\omega))), \quad \forall\ \omega \in \Omega,\ f \in C_0(\Omega, X),$$

where $u : \Omega \longrightarrow \mathcal{G}(X)$ continuous in strong operator topology and ϕ is a homeomorphism of Ω onto itself. From Lemma 2.3.1, we have

$$T^2 - (\lambda+1)T + \lambda I = 0.$$

That is,

$$u_\omega \circ u_{\phi(\omega)}(f(\phi^2(\omega))) - (\lambda+1)u_\omega(f(\phi(\omega))) + \lambda f(\omega) = 0. \tag{2.3.1}$$

Let $\omega \in \Omega$. If $\phi(\omega) \neq \omega$, then $\phi^2(\omega) = \omega$. For otherwise, there exists $h \in C_0(\Omega)$ such that $h(\omega) = 1$, $h(\phi(\omega)) = h(\phi^2(\omega)) = 0$. For $f = h \otimes x$, where x is a fixed vector in X, Equation (2.3.1) reduces to $\lambda = 0$, contradicting the assumption on λ. Now, choosing $h \in C_0(\Omega)$ such that $h(\omega) = 0$, $h(\phi(\omega)) = 1$ we get $\lambda = -1$. This implies that $u_\omega \circ u_{\phi(\omega)} = I$. Let $\phi(\omega) = \omega$ and ϕ is not the identity, then we choose an $\omega_0 \neq \phi(\omega_0)$ and conclude from above that $\lambda = -1$. This again implies that $u_\omega^2 = I$. Hence, in both cases P will be of the form $\frac{I+T}{2}$ and $T^2 = I$.

If $\phi(\omega) = \omega$ for all $\omega \in \Omega$, then we will have from Equation (2.3.1)

$$u_\omega^2 - (\lambda + 1)u_\omega + \lambda I = 0.$$

Thus from Lemma 2.3.1, there exists a projection P_ω on X such that $P_\omega + \lambda(I - P_\omega) = u_\omega$. Since u_ω is an isometry, P_ω is a GBP. Therefore, we have $Pf(\omega) = P_\omega(f(\omega))$.

This completes the proof. $\square$

Corollary 2.3.3. *Let Ω be a locally compact Hausdorff space and P a generalized bi-circular projection on $C_0(\Omega)$. Then $P = \frac{I+T}{2}$, where T is an isometry on $C_0(\Omega)$.*

We recall that the L_∞-structure of a Banach space X is the set of all projections P satisfying

$$\|x\| = \max\{\|Px\|, \|x - Px\|\} \quad \forall\, x \in X.$$

This structure is said to be trivial if it consists only of zero and the identity.

If (X_n) is a sequence of Banach spaces such that every X_n has trivial L_∞-structure, then the surjective isometries of $\bigoplus_{c_0} X_n$ is described in Theorem 1.2.17. If T is surjective isometry of $\bigoplus_{c_0} X_n$, then it is of the form

$$(Tx)_n = U_{n\pi(n)} x_{\pi(n)} \text{ for each } x = (x_n) \in \bigoplus_{c_0} X_n.$$

Here, π is a permutation of $\mathbb{N}$ and $U_{n\pi(n)}$ is a sequence of isometric operators which maps $X_{\pi(n)}$ onto X_n.

Suppose P is a GBP on $\bigoplus_{c_0} X_n$, then we have the following result.

Theorem 2.3.4. *Let P is a a generalized bi-circular projection on $\bigoplus_{c_0} X_n$ such that each X_n has trivial L_∞-structure. Then one and only one of the following holds.*

(*a*) $P = \frac{I+T}{2}$, *where T is an isometry on $\bigoplus_{c_0} X_n$.*

(*b*) $(Px)_n = P_n x_n$, *where P_n is a generalized bi-circular projection on X_n.*

Proof. Let $P + \lambda(I - P) = T$, where $\lambda \in \mathbb{T} \setminus \{1\}$ and T is an isometry on $\bigoplus_{c_0} X_n$. Then T has the form $(Tx)_n = U_{n\pi(n)} x_{\pi(n)}$ for each $x = (x_n) \in \bigoplus_{c_0} X_n$, where π and $U_{n\pi(n)}$ are as above. From Lemma 2.3.1, we have

$$T^2 - (\lambda + 1)T + \lambda I = 0$$

or for each $x = (x_n) \in \bigoplus_{c_0} X_n$

$$T^2 x - (\lambda + 1)Tx + \lambda x = 0.$$

Thus, for all $n \in \mathbb{N}$

$$U_{n\pi(n)} \circ U_{\pi(n)\pi^2(n)}\ x_{\pi^2(n)} - (\lambda + 1)U_{n\pi(n)}\ x_{\pi(n)} + \lambda x_n = 0. \tag{2.3.2}$$

For any $x = (x_n) \in \bigoplus_{c_0} X_n$, let e_{x_n} denote the vector in $\bigoplus_{c_0} X_n$ having x_n in the n^{th} coordinate and 0 elsewhere.

Let $x = (x_n) \in \bigoplus_{c_0} X_n$. If $\pi(n) \neq n$, then $\pi^2(n) = n$. Otherwise, by choosing e_{x_n} in Equation (2.3.2) we get $\lambda = 0$, which is a contradiction. Now, considering $x = e_{x_{\pi(n)}}$ Equation (2.3.2) implies that $\lambda = -1$. Thus, $U_{n\pi(n)} \circ U_{\pi(n)\pi^2(n)} = I$. If $\pi(n) = n$ and π is not the identity then by choosing $n_0 \neq \pi(n_0)$ and proceeding as above we get $\lambda = -1$ and $U_{nn}^2 = I$. Hence, assertion (a) is proved.

For the case $\pi(n) = n$ for all n, Equation (2.3.2) reduces to

$$U_{nn}^2 - (\lambda + 1)U_{nn} + \lambda I = 0.$$

Lemma 2.3.1 implies that for each $n \in \mathbb{N}$ there exists a projection P_n on X_n such that $P_n + \lambda(I - P_n) = U_{nn}$. Since U_{nn} is an isometry, P_n is a GBP and hence, $(Px)_n = P_n x_n$.

This completes the proof. □

CHAPTER 3

Projection in Convex Hull of Three Isometries

3.1 Statement of results

Let Ω be a compact connected Hausdorff space. The following are the main results of this chapter.

Theorem 3.1.1. *Let Ω be a compact connected Hausdorff space and P_0 a proper generalized 3-circular projection on $C(\Omega)$. Then there exists a surjective isometry T on $C(\Omega)$ such that*

(a) $P_0 + \omega P_1 + \omega^2 P_2 = T$, where P_1 and P_2 are as in Definition 1.2.1 and ω is a cube root of unity,

(b) $T^3 = I$.

Hence, $P_0 = \frac{I+T+T^2}{3}$.

Theorem 3.1.2. *Let Ω be a compact connected Hausdorff space. Let P be a projection on $C(\Omega)$ such that $P = \alpha_1 T_1 + \alpha_2 T_2 + \alpha_3 T_3$, where T_1, T_2, T_3 are surjective isometries of $C(\Omega)$, $\alpha_i > 0$, $i = 1, 2, 3$ and $\alpha_1 + \alpha_2 + \alpha_3 = 1$. Then either,* (*a*) $\alpha_i = \frac{1}{2}$ *for some* $i = 1, 2, 3$ $\alpha_j + \alpha_k = \frac{1}{2}$, $j, k \neq i$ *and* $T_j = T_k$, *or*

(*b*) $\alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{3}$ *and* T_1, T_2, T_3 *are distinct surjective isometries. Moreover, in this case there exists a surjective isometry* T *on* $C(\Omega)$ *such that* $T^3 = I$ *and* $P = \frac{I+T+T^2}{3}$.

All the results presented in this chapter appeared in [2].

3.2 Proof of results

Theorem 3.2.1. *Let* Ω *be a compact connected Hausdorff space and* P_0 *a generalized* 3-*circular projection on* $C(\Omega)$. *Then either,*

(*a*) λ_1 *and* λ_2 *are cube roots of unity, or*

(*b*) P_0 *is a generalized bi-circular projection. In this case,* $\lambda_1, \lambda_2 \in \{i, -i\}$.

The following two lemmas will be useful for the proof of Theorem 3.2.1 and later in Chapter 4.

Lemma 3.2.2. *Let* X *be a Banach space such that every GBP on* X *is given by* $\frac{I+L}{2}$, *where* $L \in \mathcal{G}(X)$. *Let* P_0 *be a generalized 3-circular projection on* X *and* λ_1, λ_2, P_1, P_2 *are as in Definition 1.2.1. Then* λ_1 *and* λ_2 *are of same order.*

Proof. Let $\lambda_1^m = \lambda_2^n = 1$ and $m \neq n$. Without loss of generality we assume that $m < n$. Let $P_0 + \lambda_1 P_1 + \lambda_2 P_2 = T$ where $T \in \mathcal{G}(X)$. Then $P_0 + \lambda_1^m P_1 + \lambda_2^m P_2 = (P_0 + P_1) + \lambda_2^m P_2 = T^m$. Since T^m is again a surjective isometry and $P_2 = I - (P_0 + P_1)$, by the assumption on X, we have $\lambda_2^m = -1$. Hence n divides $2m$. Similarly, we obtain $\lambda_1^n = -1$ and m divides $2n$. Thus, we have $2n = mk_1$, $2m = nk_2$. It follows that $k_1 k_2 = 4$. Since we have assumed $m < n$, this implies $k_1 = 4, k_2 = 1$. But then $-1 = \lambda_1^n = \lambda_1^{2m} = 1$ - A contradiction. Hence $m = n$. ☐

Remark 3.2.3. *Theorem 1.2.15 implies that any GBP on* $C(\Omega)$ *is of the form* $\frac{I+T}{2}$, *where* $T \in \mathcal{G}(C(\Omega))$ *and* Ω *is a compact connected Hausdorff space. Therefore, Lemma* 2.1 *in [2] follows directly from Lemma 3.2.2.*

Lemma 3.2.4. *Let X be a Banach space. Then the following assertions are equivalent.*

(i) *There exists a generalized 3-circular projection on X.*

(ii) *There exist $\lambda_1, \lambda_2 \in \mathbb{T} \setminus \{1\}$, $\lambda_1 \neq \lambda_2$ and $T \in \mathcal{G}(X)$ such that*

$$(T - I)(T - \lambda_1 I)(T - \lambda_2 I) = 0. \tag{3.2.1}$$

Proof. (i) $\Longrightarrow$ (ii) Let P_0 be a generalized 3-circular projection on X. Then there exist projections P_1, P_2, $P_0 \oplus P_1 \oplus P_2 = I$ and $\lambda_1, \lambda_2 \in \mathbb{T}\setminus\{1\}$, $\lambda_1 \neq \lambda_2$ such that $P_0 + \lambda_1 P_1 + \lambda_2 P_2$ is a surjective isometry. Let $P_0 + \lambda_1 P_1 + \lambda_2 P_2 = T$. By eliminating P_1 and P_2 we get

$$P_0 = \frac{(T - \lambda_1 I)(T - \lambda_2 I)}{(1 - \lambda_1)(1 - \lambda_2)}.$$

Thus, we have

$$(T - I)(T - \lambda_1 I)(T - \lambda_2 I) =$$

$$[(\lambda_1 - 1)P_1 + (\lambda_2 - 1)P_2](1 - \lambda_1)(1 - \lambda_2)P_0 = 0.$$

(ii) $\Longrightarrow$ (i) We define $P_0 = \dfrac{(T - \lambda_1 I)(T - \lambda_2 I)}{(1 - \lambda_1)(1 - \lambda_2)}$,

$$P_1 = \frac{(T - \lambda_2 I)(T - I)}{(\lambda_1 - 1)(\lambda_1 - \lambda_2)} \text{ and } P_2 = \frac{(T - \lambda_1 I)(T - I)}{(\lambda_2 - 1)(\lambda_2 - \lambda_1)}.$$

It is easy to check $P_0 + \lambda_1 P_1 + \lambda_2 P_2 = T$ and $P_0 + P_1 + P_2 = I$. Let

$$S = \frac{(T - \lambda_1 I)(T - \lambda_2 I)}{(1 - \lambda_1)^2(1 - \lambda_2)^2}.$$

Then we have

$$\begin{aligned}
P_0^2 &= \frac{(T - \lambda_1 I)(T - \lambda_2 I)}{(1 - \lambda_1)^2(1 - \lambda_2)^2}[T^2 - (\lambda_1 + \lambda_2)T + \lambda_1\lambda_2 I] \\
&= S\,[T^2 - (\lambda_1 + \lambda_2)(T - I) - (\lambda_1 + \lambda_2)I + \lambda_1\lambda_2 I] \\
&= S\,[T^2 - (\lambda_1 + \lambda_2)I + \lambda_1\lambda_2 I] \quad (\text{Eq. (3.2.1)} \Rightarrow S(T - I) = 0) \\
&= S\,[T^2 - I + I - (\lambda_1 + \lambda_2)I + \lambda_1\lambda_2 I] \\
&= S\,[I - (\lambda_1 + \lambda_2)I + \lambda_1\lambda_2 I] \quad (\because S(T^2 - I) = 0) \\
&= S\,[(1 - \lambda_1)(1 - \lambda_2)I]
\end{aligned}$$

$$= \quad P_0.$$

Similarly, we can show that P_1 and P_2 are projections. Also, Equation (3.2.1) implies that $P_iP_j = 0$ for $i \neq j$. Hence, P_0 is a generalized 3-circular projection and assertion (i) is proved. □

Proof of Theorem 3.2.1

Let P_0 be a generalized 3-circular projection. Then there exist projections P_1, P_2, $P_0 \oplus P_1 \oplus P_2 = I$ and $\lambda_1, \lambda_2 \in \mathbb{T} \setminus \{1\}$, $\lambda_1 \neq \lambda_2$ such that $P_0 + \lambda_1 P_1 + \lambda_2 P_2 = T$, for some $T \in \mathcal{G}(C(\Omega))$. By Theorem 1.2.10, there exist a homeomorphism ϕ on Ω and a continuous function $u : \Omega \to \mathbb{T}$ such that for any $f \in C(\Omega),\ \omega \in \Omega$ we have $Tf(\omega) = u(\omega)f(\phi(\omega))$. From Lemma 3.2.4, we have

$$P_0 = \frac{(T - \lambda_1 I)(T - \lambda_2 I)}{(1 - \lambda_1)(1 - \lambda_2)}. \tag{3.2.2}$$

Also, if we take $\lambda_1 + \lambda_2 = a$ and $\lambda_1 \lambda_2 = b$, Equation (3.2.1) implies that

$$T^3 - (1 + a)T^2 + (a + b)T - bI = 0$$

or

$$u(\omega)u(\phi(\omega))u(\phi^2(\omega))f(\phi^3(\omega)) - (1 + a)u(\omega)u(\phi(\omega))f(\phi^2(\omega))$$
$$+ (a + b)u(\omega)f(\phi(\omega)) - bf(\omega) = 0. \tag{3.2.3}$$

If ω, $\phi(\omega)$, $\phi^2(\omega)$ and $\phi^3(\omega)$ are all distinct, then we choose $f \in C(\Omega)$ such that $f(\phi(\omega)) = f(\phi^2(\omega)) = f(\phi^3(\omega)) = 0$ and $f(\omega) = 1$ to get $b = 0$, which is a contradiction.

So, we consider the following cases:

(I) $\omega = \phi^2(\omega),\ \omega \neq \phi(\omega)$. Then we have $\phi(\omega) = \phi^3(\omega)$. We consider a function $f \in C(\Omega)$ such that $f(\omega) = 1,\ f(\phi(\omega)) = 0$. Then Equation (3.2.3) becomes

$$-(1 + a)u(\omega)u(\phi(\omega)) - b = 0, \text{ hence } u(\omega)u(\phi(\omega)) = -\frac{b}{1 + a}.$$

Similarly, considering an $f \in C(\Omega)$ such that $f(\omega) = 0,\ f(\phi(\omega)) = 1$, Equation (3.2.3) gives

$$u(\omega)u(\phi(\omega)) = -(a + b). \text{ Thus, we have } \frac{b}{1 + a} = a + b.$$

That is,

$$(1+\lambda_1+\lambda_2)(\lambda_1+\lambda_2+\lambda_1\lambda_2)=\lambda_1\lambda_2,$$

or

$$2+\lambda_1+\lambda_2+\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\frac{\lambda_1}{\lambda_2}+\frac{\lambda_2}{\lambda_1}=0.$$

By Lemma 3.2.2, there exists an n such that both λ_1 and λ_2 are n^{th} roots of identity. Hence, we may assume $\lambda_2=\lambda_1^m$ for some m.

Thus, the above equation can be written as,

$$\lambda_1^{2m}+\lambda_1^{2m-1}+\lambda_1^{m+1}+2\lambda_1^m+\lambda_1^{m-1}+\lambda_1+1=0,$$

or

$$(\lambda_1+1)(\lambda_1^{m-1}+1)(\lambda_1^m+1)=0.$$

Therefore, we have $\lambda_1=-1$, $\lambda_1^m=-1$ or $\lambda_1^{m-1}=-1$.

If $\lambda_1=-1$, then from Lemma 3.2.2 we have $\lambda_2=\pm 1$. Since $\lambda_2\neq 1$, we get $\lambda_2=-1$. Therefore, $\lambda_1=\lambda_2$ - A contradiction on the assumption on λ_1 and λ_2.

If $\lambda_1^m=-1$, then we have $\lambda_2=-1$ and by the same argument above we get $\lambda_1=-1$ which is a contradiction again.

If $\lambda_1^{m-1}=-1$ then we have $\lambda_2=\lambda_1^m=-\lambda_1$. It follows that, $T=P_0+\lambda_1(P_1-P_2)$. This implies that $T^2=P_0+\lambda_1^2(P_1+P_2)$. Since T^2 is an isometry, P_0 is a GBP and hence by Theorem 1.2.15, we have $\lambda_1^2=-1$. Therefore, we get $\lambda_1=\pm i$ and $\lambda_2=\mp i$.

Thus, assertion (b) is proved.

(II) $\omega=\phi^3(\omega)$, $\omega\neq\phi(\omega)\neq\phi^2(\omega)\neq\omega$. By choosing an $f\in C(\Omega)$ such that $f(\phi(\omega))=1,\ f(\omega)=f(\phi^2(\omega))=0$, Equation (3.2.3) implies that $a+b=0$. Similarly, if we choose an $f\in C(\Omega)$ such that $f(\phi^2(\omega))=1,\ f(\omega)=f(\phi(\omega))=0$ we get $1+a=0$. Thus, we have $a=-1$ and $b=1$. This implies that λ_1 and λ_2 are the cube roots of identity and hence assertion (a) is proved.

(III) $\omega=\phi(\omega)$. In this case, Equation (3.2.3) gives $u^3(\omega)-(1+a)u^2(\omega)+(a+b)u(\omega)-b=0$. Thus, for each $\omega\in\Omega$, $u(\omega)$ has 3 possible values. Now, if $\omega=\phi(\omega)$ is the entire set then from connectedness of Ω it follows that u is a constant function. By Equation (3.2.2),

P_0 is constant multiple of the identity operator and since P_0 is a projection, it is either I or 0 operator.

This completes the proof of Theorem 3.2.1. $\square$

Proof of Theorem 3.1.1

If P_0 is a proper generalized 3-circular projection on $C(\Omega)$, then it is not a GBP. Hence, by Theorem 3.2.1 we conclude that λ_1 and λ_2 are cube roots of unity. This completes the proof. $\square$

Proof of Theorem 3.1.2

We start by observing the following fact. If P is a proper projection, then $\exists\ f \in C(\Omega),\ f \neq 0$ such that $Pf = 0$. Hence, $\alpha_1 T_1 f + \alpha_2 T_2 f = -\alpha_3 T_3 f$. Since T_1, T_2 and T_3 are isometries, by taking norms we have $\alpha_1 + \alpha_2 \geq \alpha_3$. Similarly, $\alpha_2 + \alpha_3 \geq \alpha_1$ and $\alpha_1 + \alpha_3 \geq \alpha_2$. Thus, if P is a proper projection then α_1, α_2 and α_3 are the lengths of sides of a triangle. It is also evident that $\alpha_i \leq 1/2$, $i = 1, 2, 3$.

Let $T_i f(\omega) = u_i(\omega) f(\phi_i(\omega)),\ i = 1, 2, 3$ where u_i is a continuous function from Ω to $\mathbb{T}$ and ϕ_i is a homeomorphism on Ω.

P is a projection if and only if

$$\alpha_1 u_1(\omega)[\alpha_1 u_1(\phi_1(\omega)) f(\phi_1^2(\omega)) + \alpha_2 u_2(\phi_1(\omega)) f(\phi_2 \circ \phi_1(\omega)) + \alpha_3 u_3(\phi_1(\omega)) f(\phi_3 \circ \phi_1(\omega))] +$$

$$\alpha_2 u_2(\omega)[\alpha_1 u_1(\phi_2(\omega)) f(\phi_1 \circ \phi_2(\omega)) + \alpha_2 u_2(\phi_2(\omega)) f(\phi_2^2(\omega)) + \alpha_3 u_3(\phi_2(\omega)) f(\phi_3 \circ \phi_2(\omega))] +$$

$$\alpha_3 u_3(\omega)[\alpha_1 u_1(\phi_3(\omega)) f(\phi_1 \circ \phi_3(\omega)) + \alpha_2 u_2(\phi_3(\omega)) f(\phi_2 \circ \phi_3(\omega)) + \alpha_3 u_3(\phi_3(\omega)) f(\phi_3^2(\omega))] =$$

$$\alpha_1 u_1(\omega) f(\phi_1(\omega)) + \alpha_2 u_2(\omega) f(\phi_2(\omega)) + \alpha_3 u_3(\omega) f(\phi_3(\omega)). \qquad (**)$$

We partition Ω as follows:

$A = \{\omega \in \Omega :\ \phi_1(\omega) = \phi_2(\omega) = \phi_3(\omega)\}$,

$B_i = \{\omega \in \Omega :\ \omega = \phi_j(\omega) = \phi_k(\omega) \neq \phi_i(\omega)\}$,

$C_i = \{\omega \in \Omega :\ \omega = \phi_i(\omega) \neq \phi_j(\omega) = \phi_k(\omega)\}$,

$D_i = \{\omega \in \Omega :\ \omega = \phi_i(\omega) \neq \phi_j(\omega) \neq \phi_k(\omega) \neq \omega\}$,

$E_i = \{\omega \in \Omega :\ \omega \neq \phi_i(\omega) \neq \phi_j(\omega) = \phi_k(\omega) \neq \omega\}$ and

$F = \{\omega \in \Omega :$ none of $\omega,\ \phi_1(\omega),\ \phi_2(\omega),\ \phi_3(\omega)$ are equal$\}$,

where $i, j, k = 1, 2, 3$.

Since the proof is long, we divide the proof into the following steps.

Step I. We show that $\Omega \neq A$ (Lemma 3.2.5).

Step II. We show that if $\Omega = B_i$, $i = 1, 2, 3$ then $\alpha_i = 1/2$ (Lemma 3.2.6).

Step III. We show that $E_i = F = \emptyset$, for $i = 1, 2, 3$ (Lemma 3.2.7).

Step IV. We show that if $\omega \in C_i$, $i = 1, 2, 3$ then $\alpha_i = 1/2$ (Lemma 3.2.8).

Step V. We show that if $\omega \in D_i$, $i = 1, 2, 3$ then $\alpha_1 = \alpha_2 = \alpha_3 = 1/3$ (Lemma 3.2.9).

Step VI. This is the final step. In this step we will show that only certain combinations of B_i, C_i and D_i are allowed (Lemma 3.2.10), $i = 1, 2, 3$. Then we will complete the proof.

Lemma 3.2.5. *$\Omega \neq A$.*

Proof. Suppose $A \neq \emptyset$ and $\omega \in A$. Then we have $\phi_1(\omega) = \phi_2(\omega) = \phi_3(\omega)$. Equation $(**)$ is reduced to

$$[\alpha_1 u_1(\omega) + \alpha_2 u_2(\omega) + \alpha_3 u_3(\omega)][\alpha_1 u_1(\phi_1(\omega)) f(\phi_1^2(\omega)) + \alpha_2 u_2(\phi_1(\omega)) f(\phi_2^2(\omega)) +$$

$$\alpha_3 u_3(\phi_1(\omega)) f(\phi_3^2(\omega))] = [\alpha_1 u_1(\omega) + \alpha_2 u_2(\omega) + \alpha_3 u_3(\omega)] f(\phi_1(\omega)). \tag{3.2.4}$$

Let $A_1 = \{\omega \in A : \alpha_1 u_1(\omega) + \alpha_2 u_2(\omega) + \alpha_3 u_3(\omega) \neq 0\}$ and $A_2 = A \setminus A_1$.

If $\omega \in A_1$, then we have

$$\alpha_1 u_1(\phi_1(\omega)) f(\phi_1^2(\omega)) + \alpha_2 u_2(\phi_1(\omega)) f(\phi_2^2(\omega)) + \alpha_3 u_3(\phi_1(\omega)) f(\phi_3^2(\omega)) = f(\phi_1(\omega)).$$

First evaluating at the constant function 1 we observe that

$$\alpha_1 u_1(\phi_1(\omega)) + \alpha_2 u_2(\phi_1(\omega)) + \alpha_3 u_3(\phi_1(\omega)) = 1.$$

Hence, we get $u_i(\phi_i(\omega)) = 1,\ i = 1, 2, 3$. Thus we obtain,

$$\alpha_1 f(\phi_1^2(\omega)) + \alpha_2 f(\phi_2^2(\omega)) + \alpha_3 f(\phi_3^2(\omega)) = f(\phi_1(\omega)).$$

Now, if $\phi_1(\omega)$ is not equal to any of $\phi_i^2(\omega),\ i = 1, 2, 3$ then choosing an $f \in C(\Omega)$ such that $f(\phi_1(\omega)) = 1$ and $f(\phi_i^2(\omega) = 0$, we get a contradiction. Similarly, if $\phi_1(\omega)$ is equal to one or two among $\phi_i^2(\omega)$, $i = 1, 2, 3$ then choosing an appropriate f we get either $\alpha_i = 1$ or $\alpha_j + \alpha_k = 1$, both contradicting the choices of $\alpha_1, \alpha_2, \alpha_3$ where $j, k = 1, 2, 3$.

Thus in this case, we must have, $\phi_1^2(\omega) = \phi_2^2(\omega) = \phi_3^2(\omega) = \phi_1(\omega)$ or $\omega = \phi_1(\omega) = \phi_2(\omega) = \phi_3(\omega)$. Hence, $Pf(\omega) = f(\omega)$ if $\omega \in A_1$ and $Pf(\omega) = 0$ if $\omega \in A_2$. In particular, for the constant function $1, P1$ is a $0, 1$ valued function. By the connectedness of Ω we have $\Omega \neq A$. □

Lemma 3.2.6. *If* $\Omega = B_i$, *then* $\alpha_i = 1/2$ *and* $u_j(\omega) = u_k(\omega) = u_j(\phi_i(\omega)) = u_k(\phi_i(\omega)) = u_i(\omega)u_i(\phi_i(\omega)) = 1$, *where* $i, j, k = 1, 2, 3$.

Proof. Let us consider any $\omega \in B_1$, that is, $\omega = \phi_3(\omega) = \phi_2(\omega) \neq \phi_1(\omega)$. The case of $\omega \in B_2$ or B_3 is similar. Equation $(**)$ is reduced to

$$\alpha_1 u_1(\omega)[\alpha_1 u_1(\phi_1(\omega)) f(\phi_1^2(\omega)) + \alpha_2 u_2(\phi_1(\omega)) f(\phi_2 \circ \phi_1(\omega)) + \alpha_3 u_3(\phi_1(\omega)) f(\phi_3 \circ \phi_1(\omega))]$$

$$+ [\alpha_2 u_2(\omega) + \alpha_3 u_3(\omega)]\{\alpha_1 u_1(\omega) f(\phi_1(\omega)) + [\alpha_2 u_2(\omega) + \alpha_3 u_3(\omega)] f(\omega)\}$$

$$= \alpha_1 u_1(\omega) f(\phi_1(\omega)) + [\alpha_2 u_2(\omega) + \alpha_3 u_3(\omega)] f(\omega). \tag{3.2.5}$$

We claim that $\alpha_2 u_2(\omega) + \alpha_3 u_3(\omega) \neq 0$. Suppose on the contrary that $\alpha_2 u_2(\omega) + \alpha_3 u_3(\omega) = 0$. Then we get $\alpha_2 = \alpha_3$, $u_2(\omega) + u_3(\omega) = 0$ and Equation (3.2.5) becomes

$$\alpha_1 u_1(\phi_1(\omega)) f(\phi_1^2(\omega)) + \alpha_2 u_2(\phi_1(\omega)) f(\phi_2 \circ \phi_1(\omega)) + \alpha_3 u_3(\phi_1(\omega)) f(\phi_3 \circ \phi_1(\omega)) = f(\phi_1(\omega)).$$

By the same argument applied in proof of Lemma 3.2.5, we conclude that $\phi_1(\omega) = \phi_1^2(\omega) = \phi_2 \circ \phi_1(\omega) = \phi_3 \circ \phi_1(\omega)$, which contradicts the choice of ω.

We now choose a function $f \in C(\Omega)$ such that $f(\omega) = 1$, $f(\phi_1(\omega)) = f(\phi_2 \circ \phi_1(\omega)) = f(\phi_3 \circ \phi_1(\omega)) = 0$. So, Equation (3.2.5) becomes

$$\alpha_1^2 u_1(\omega) u_1(\phi_1(\omega)) f(\phi_1^2(\omega)) + [\alpha_2 u_2(\omega) + \alpha_3 u_3(\omega)]^2 = \alpha_2 u_2(\omega) + \alpha_3 u_3(\omega). \tag{3.2.6}$$

We observe that $\phi_1^2(\omega)$ must be equal to one of ω, $\phi_2 \circ \phi_1(\omega)$ or $\phi_3 \circ \phi_1(\omega)$. If $\phi_1^2(\omega) = \phi_3 \circ \phi_1(\omega)$ or $\phi_2 \circ \phi_1(\omega)$, then we have $f(\phi_1^2(\omega)) = 0$. This implies that $\alpha_2 u_2(\omega) + \alpha_3 u_3(\omega) = 1$

as $\alpha_2 u_2(\omega) + \alpha_3 u_3(\omega) \neq 0$. It follows that, $1 \leq \alpha_2 + \alpha_3$, a contradiction to the fact that $\alpha_1 + \alpha_2 + \alpha_3 = 1$. Therefore, we get $\phi_1^2(\omega) = \omega$ and Equation (3.2.6) is reduced to

$$\alpha_1^2 u_1(\omega) u_1(\phi_1(\omega)) + [\alpha_2 u_2(\omega) + \alpha_3 u_3(\omega)]^2 = \alpha_2 u_2(\omega) + \alpha_3 u_3(\omega). \tag{3.2.7}$$

Again, for a function $f \in C(\Omega)$ such that $f(\omega) = 0$, $f(\phi_1(\omega)) = 1$, Equation (3.2.5) reduces to

$$\alpha_2 u_2(\omega) + \alpha_3 u_3(\omega) + \alpha_2 u_2(\phi_1(\omega)) f(\phi_2 \circ \phi_1(\omega)) + \alpha_3 u_3(\phi_1(\omega)) f(\phi_3 \circ \phi_1(\omega)) = 1. \tag{3.2.8}$$

By a similar line of argument, we conclude that $\phi_1(\omega) = \phi_2 \circ \phi_1(\omega) = \phi_3 \circ \phi_1(\omega)$. So, Equation (3.2.8) becomes

$$\alpha_2 u_2(\omega) + \alpha_3 u_3(\omega) + \alpha_2 u_2(\phi_1(\omega)) + \alpha_3 u_3(\phi_1(\omega)) = 1. \tag{3.2.9}$$

Now, we have

$$Pf(\omega) = \alpha_1 u_1(\omega) f(\phi_1(\omega)) + [\alpha_2 u_2(\omega) + \alpha_3 u_3(\omega)] f(\omega),$$

which implies that

$$|Pf(\omega)| \leq |\alpha_2 u_2(\omega) + \alpha_3 u_3(\omega)||f(\omega)| + \alpha_1 |f(\phi_1(\omega))|.$$

As $\Omega = B_1$, there exist $\omega_0 \in \Omega$ and $f \in C(\Omega)$ such that $||f|| = 1 = |Pf(\omega_0)|$. This shows that $|\alpha_2 u_2(\omega_0) + \alpha_3 u_3(\omega_0)| = \alpha_2 + \alpha_3$. Therefore, we have $u_2(\omega_0) = u_3(\omega_0) = 1$. From Equation (3.2.7) we get $\alpha_1 \geq 1/2$. Since $\alpha_1 \leq 1/2$, we conclude that $\alpha_1 = 1/2$. Also, from Equation (3.2.9) we have $u_2(\omega) = u_3(\omega) = u_2(\phi_1(\omega)) = u_3(\phi_1(\omega)) = 1$. Moreover, Equation (3.2.7) implies that $u_1(\omega) u_1(\phi_(\omega)) = 1$. □

Lemma 3.2.7. *For $i = 1, 2, 3$ $E_i = \emptyset$ and $F = \emptyset$.*

Proof. We show $E_1 = \emptyset$. For the case of E_2 and E_3, the proof is exactly the same.

Let $\omega \in E_1$, that is, $\omega \neq \phi_1(\omega) \neq \phi_2(\omega) = \phi_3(\omega) \neq \omega$. Then Equation $(**)$ reduces to

$$\alpha_1 u_1(\omega)[\alpha_1 u_1(\phi_1(\omega))f(\phi_1^2(\omega)) + \alpha_2 u_2(\phi_1(\omega))f(\phi_2 \circ \phi_1(\omega)) + \alpha_3 u_3(\phi_1(\omega))f(\phi_3 \circ \phi_1(\omega))] +$$

$$[\alpha_2 u_2(\omega)+\alpha_3 u_3(\omega)][\alpha_1 u_1(\phi_2(\omega))f(\phi_1 \circ \phi_2(\omega))+\alpha_2 u_2(\phi_2(\omega))f(\phi_2^2(\omega))+\alpha_3 u_3(\phi_2(\omega))f(\phi_3^2(\omega))]$$

$$= \alpha_1 u_1(\omega)f(\phi_1(\omega)) + [\alpha_2 u_2(\omega) + \alpha_3 u_3(\omega)]f(\phi_2(\omega)). \tag{3.2.10}$$

First we claim $\alpha_2 u_2(\omega) + \alpha_3 u_3(\omega) \neq 0$. To see the claim, suppose $\alpha_2 u_2(\omega) + \alpha_3 u_3(\omega) = 0$. Then Equation (3.2.10) further reduces to

$$\alpha_1 u_1(\phi_1(\omega))f(\phi_1^2(\omega)) + \alpha_2 u_2(\phi_1(\omega))f(\phi_2 \circ \phi_1(\omega)) + \alpha_3 u_3(\phi_1(\omega))f(\phi_3 \circ \phi_1(\omega)) = f(\phi_1(\omega)).$$

By similar argument which was applied in the proof of Lemma 3.2.5, we get $\phi_1(\omega) = \phi_3 \circ \phi_1(\omega) = \phi_2 \circ \phi_1(\omega) = \phi_1^2(\omega)$, which is clearly a contradiction to the choice of ω.

Secondly, we choose a function $f \in C(\Omega)$ such that $f(\phi_1(\omega)) = 1$ and $f(\phi_2(\omega)) = f(\phi_1 \circ \phi_2(\omega)) = f(\phi_1^2(\omega)) = 0$. Equation (3.2.10) now reduces to

$$\alpha_1 u_1(\omega)[\alpha_2 u_2(\phi_1(\omega))f(\phi_2 \circ \phi_1(\omega)) + \alpha_3 u_3(\phi_1(\omega))f(\phi_3 \circ \phi_1(\omega))] +$$

$$[\alpha_2 u_2(\omega) + \alpha_3 u_3(\omega)][\alpha_2 u_2(\phi_2(\omega))f(\phi_2^2(\omega)) + \alpha_3 u_3(\phi_2(\omega))f(\phi_3^2(\omega))] = \alpha_1 u_1(\omega) \tag{3.2.11}$$

If $\phi_1(\omega)$ is not equal to any of the points $\phi_2 \circ \phi_1(\omega), \phi_3 \circ \phi_1(\omega), \phi_2^2(\omega)$ and $\phi_3^2(\omega)$, then we could have chosen our f to have value 0 at these points and this would have lead us to a contradiction. If $\phi_1(\omega) = \phi_2 \circ \phi_1(\omega)$, then clearly we could choose $f(\phi_2^2(\omega)) = 0$. If both $\phi_3 \circ \phi_1(\omega)$ and $\phi_3^2(\omega)$ are not equal to $\phi_1(\omega)$, then choosing f to take value 0 at $\phi_3 \circ \phi_1(\omega)$ and $\phi_3^2(\omega)$ we have

$$\alpha_1 \alpha_2 u_1(\omega) u_2(\phi_1(\omega)) = \alpha_1 u_1(\omega)$$

and hence we get $\alpha_2 = 1$, a contradiction again. Thus, either of $\phi_3 \circ \phi_1(\omega)$ and $\phi_3^2(\omega)$ is equal to $\phi_1(\omega)$.

Similar consideration with $\phi_1(\omega) = \phi_3 \circ \phi_1(\omega)$, $\phi_1(\omega) = \phi_2^2(\omega)$ and $\phi_1(\omega) = \phi_3^2(\omega)$ lead us to the conclusion that $\phi_1(\omega)$ will be equal to exactly two elements of the set

$$\{\phi_2 \circ \phi_1(\omega),\ \phi_3 \circ \phi_1(\omega),\ \phi_2^2(\omega),\ \phi_3^2(\omega)\}.$$

If $\phi_1(\omega) = \phi_2 \circ \phi_1(\omega) = \phi_3 \circ \phi_1(\omega)$ then Equation (3.2.11) will imply that $\alpha_2 u_2(\phi_1(\omega)) + \alpha_3 u_3(\phi_1(\omega)) = 1$ - A contradiction.

Finally, let us suppose that $\phi_1(\omega) = \phi_2 \circ \phi_i(\omega) = \phi_3 \circ \phi_j(\omega)$ where $i = 1, 2,\ j = 2, 3;$ $i \neq j$ and choose a function f such that $f(\phi_2(\omega)) = 1$ and $f(\phi_1(\omega)) = f(\phi_2 \circ \phi_{i_1}(\omega)) = f(\phi_3 \circ \phi_{j_1}(\omega)) = 0$, where $i_1 \neq i,\ j_1 \neq j$, and $i_1 = 1, 2;\ j = 1, 3$. So, Equation (3.2.10) becomes

$$\begin{aligned}\alpha_1^2 u_1(\omega) u_1(\phi_1(\omega)) f(\phi_1^2(\omega)) + [\alpha_2 u_2(\omega) + \alpha_3 u_3(\omega)]\alpha_1 u_1(\phi_2(\omega)) f(\phi_1 \circ \phi_2(\omega))& \\ = \alpha_2 u_2(\omega) + \alpha_3 u_3(\omega).& \qquad (3.2.12)\end{aligned}$$

If $\phi_2(\omega)$ is not equal to any one of $\phi_1^2(\omega)$ or $\phi_1 \circ \phi_2(\omega)$, then we can choose f to be 0 at $\phi_1^2(\omega)$ and $\phi_1 \circ \phi_2(\omega)$, and this will imply that $\alpha_2 u_2(\omega) + \alpha_3 u_3(\omega) = 0$, which is a contradiction. If $\phi_2(\omega) = \phi_1 \circ \phi_2(\omega)$, then by choosing f to be 0 at $\phi_1^2(\omega)$ we will get $\alpha_1 = 1$ which is a contradiction again. Therefore, we have $\phi_2(\omega) = \phi_1^2(\omega)$. Similarly, $\phi_1 \circ \phi_2(\omega)$ must be equal to at least one of $\phi_2 \circ \phi_{i_1}(\omega)$ or $\phi_2 \circ \phi_{j_1}(\omega)$. But in this case we will be left with 3 or 4 distinct points in Equation (3.2.10). By choosing f to be 0 at $\phi_1(\omega)$ and $\phi_2(\omega)$ and large enough at other points on the left hand side of Equation (3.2.10) we will get a contradiction.

This shows that $E_1 = \emptyset$.

We now show that $F = \emptyset$.

Suppose $F \neq \emptyset$. We choose $\omega \in F$, then we have $\omega,\ \phi_1(\omega),\ \phi_2(\omega),\ \phi_3(\omega)$ are all distinct.

Let us consider the following matrix:

$$\begin{pmatrix} \phi_1(\omega) & \phi_2(\omega) & \phi_3(\omega) \\ \phi_1^2(\omega) & \phi_2 \circ \phi_1(\omega) & \phi_3 \circ \phi_1(\omega) \\ \phi_1 \circ \phi_2(\omega) & \phi_2^2(\omega) & \phi_3 \circ \phi_2(\omega) \\ \phi_1 \circ \phi_3(\omega) & \phi_2 \circ \phi_3(\omega) & \phi_3^2(\omega) \end{pmatrix}$$

We observe that points belonging to any column are all non equal. We choose f such that $f(\phi_1(\omega)) = 1$ and $f(\phi_2(\omega)) = f(\phi_3(\omega)) = f(\phi_1^2(\omega)) = f(\phi_1 \circ \phi_2(\omega)) = f(\phi_1 \circ \phi_3(\omega)) = 0$. Equation $(**)$ becomes

$$\alpha_1 u_1(\omega)[\alpha_2 u_2(\phi_1(\omega)) f(\phi_2 \circ \phi_1(\omega)) + \alpha_3 u_3(\phi_1(\omega)) f(\phi_3 \circ \phi_1(\omega))] +$$

$$\alpha_2 u_2(\omega)[\alpha_2 u_2(\phi_2(\omega)) f(\phi_2^2(\omega)) + \alpha_3 u_3(\phi_2(\omega)) f(\phi_3 \circ \phi_2(\omega))] +$$

$$\alpha_3 u_3(\omega)[\alpha_2 u_2(\phi_3(\omega)) f(\phi_2 \circ \phi_3(\omega)) + \alpha_3 u_3(\phi_3(\omega)) f(\phi_3^2(\omega))] = \alpha_1 u_1(\omega) f(\phi_1(\omega)). \quad (3.2.13)$$

Equation (3.2.13) implies that $\phi_1(\omega)$ must be equal to at least 2 elements from the set

$$\{\phi_2 \circ \phi_1(\omega),\ \phi_3 \circ \phi_1(\omega),\ \phi_2^2(\omega),\ \phi_3 \circ \phi_2(\omega),\ \phi_2 \circ \phi_3(\omega),\ \phi_3^2(\omega)\}.$$

Since this set does not contain three equal elements, it follows that $\phi_1(\omega)$ is equal to exactly two; say $\phi_2 \circ \phi_{i_1}(\omega)$ and $\phi_3 \circ \phi_{j_1}(\omega)$ with $i_1, j_1 \in \{1, 2, 3\}$. Therefore, we have

$$\alpha_{i_1} \alpha_2 u_{i_1}(\omega) u_2(\phi_{i_1}(\omega)) + \alpha_{j_1} \alpha_3 u_{j_1}(\omega) u_3(\phi_{j_1}(\omega)) = \alpha_1 u_1(\omega).$$

This implies that

$$\alpha_1 \leq \alpha_2 \alpha_{i_1} + \alpha_3 \alpha_{j_1}.$$

Similar arguments applied to $\phi_2(\omega)$ and $\phi_3(\omega)$ imply the inequalities:

$$\alpha_2 \leq \alpha_1 \alpha_{i_2} + \alpha_3 \alpha_{j_2} \text{ and } \alpha_3 \leq \alpha_1 \alpha_{i_3} + \alpha_2 \alpha_{j_3}.$$

Adding these three inequalities we get

$$\begin{aligned} 1 = \alpha_1 + \alpha_2 + \alpha_3 &\leq \alpha_1(\alpha_{i_2} + \alpha_{i_3}) + \alpha_2(\alpha_{i_1} + \alpha_{j_3}) + \alpha_3(\alpha_{j_1} + \alpha_{j_2}) \\ &\leq \max\{\alpha_{i_2} + \alpha_{i_3},\ \alpha_{i_1} + \alpha_{j_3},\ \alpha_{j_1} + \alpha_{j_2}\}. \end{aligned}$$

This is impossible. Hence, we have $F = \emptyset$. $\square$

We now set ourselves to show the following.

Lemma 3.2.8. *If* $\omega \in C_i$, $i = 1, 2, 3$ *then* $\alpha_i = 1/2$; $u_i(\omega) = u_i(\phi_j(\omega)) = 1$, $u_j(\omega) = u_k(\omega)$, $u_j(\phi_j(\omega)) = u_k(\phi_j(\omega))$ *and* $u_j(\omega)u_j(\phi_j(\omega)) = 1$ *for* $j, k = 1, 2, 3$; $i \neq j, k$.

Proof. We prove the result for $i = 1$. For $i = 2$ and 3 the argument is similar.

Let $\omega \in C_1$, that is, $\omega = \phi_1(\omega) \neq \phi_2(\omega) = \phi_3(\omega)$, then Equation $(**)$ reduces to

$$\alpha_1 u_1(\omega)[\alpha_1 u_1(\omega))f(\omega) + \alpha_2 u_2(\omega) f(\phi_2(\omega)) + \alpha_3 u_3(\omega) f(\phi_2(\omega))] +$$

$$[\alpha_2 u_2(\omega)+\alpha_3 u_3(\omega)][\alpha_1 u_1(\phi_2(\omega)) f(\phi_1\circ\phi_2(\omega))+\alpha_2 u_2(\phi_2(\omega)) f(\phi_2^2(\omega))+\alpha_3 u_3(\phi_2(\omega)) f(\phi_3^2(\omega))]$$

$$= \alpha_1 u_1(\omega) f(\omega) + [\alpha_2 u_2(\omega) + \alpha_3 u_3(\omega)] f(\phi_2(\omega)). \tag{3.2.14}$$

We note that in this case we must have $\alpha_2 u_2(\omega)+\alpha_3 u_3(\omega) \neq 0$; otherwise Equation (3.2.14) will give us $\alpha_1 = 1$.

We choose a function $f \in C(\Omega)$ such that $f(\phi_2(\omega)) = 1$, $f(\omega) = f(\phi_2^2(\omega)) = f(\phi_3^2(\omega)) = 0$ which will reduce Equation (3.2.14) to

$$\alpha_1 u_1(\omega)[\alpha_2 u_2(\omega) + \alpha_3 u_3(\omega)] + [\alpha_2 u_2(\omega) + \alpha_3 u_3(\omega)]\alpha_1 u_1(\phi_2(\omega)) f(\phi_1 \circ \phi_2(\omega))$$

$$= \alpha_2 u_2(\omega) + \alpha_3 u_3(\omega). \tag{3.2.15}$$

Since $\alpha_2 u_2(\omega) + \alpha_3 u_3(\omega) \neq 0$, we obtain

$$\alpha_1 u_1(\omega) + \alpha_1 u_1(\phi_2(\omega)) f(\phi_1 \circ \phi_2(\omega)) = 1.$$

This implies that $\phi_1 \circ \phi_2(\omega) = \phi_2(\omega)$ and $\alpha_1 \geq 1/2$. Since $\alpha_i \leq 1/2$, $\forall i$ we conclude $\alpha_1 = 1/2$ and $u_1(\omega) = u_1(\phi_2(\omega)) = 1$.

Choosing a function f such that $f(\omega) = f(\phi_2(\omega)) = 0$, Equation (3.2.14) becomes

$$\alpha_2 u_2(\phi_2(\omega)) f(\phi_2^2(\omega)) + \alpha_3 u_3(\phi_2(\omega)) f(\phi_3^2(\omega)) = 0.$$

The points $\phi_2^2(\omega)$ and $\phi_3^2(\omega)$ must be equal to one of ω or $\phi_2(\omega)$. Since $\phi_2^2(\omega)$ and $\phi_3^2(\omega)$ cannot be equal to $\phi_2(\omega)$, we have $\phi_2^2(\omega) = \phi_3^2(\omega) = \omega$. We now choose a function f such that $f(\omega) = 1$, $f(\phi_2(\omega) = 0$, Equation (3.2.14) is reduced to

$$[\alpha_2 u_2(\omega) + \alpha_3 u_3(\omega)][\alpha_2 u_2(\phi_2(\omega)) + \alpha_3 u_3(\phi_2(\omega))] = \frac{1}{4},$$

or

$$\alpha_2^2 u_2(\omega)u_2(\phi_2(\omega)) + \alpha_2\alpha_3 u_2(\omega)u_3(\phi_2(\omega)) + \alpha_2\alpha_3 u_3(\omega)u_2(\phi_2(\omega)) + \alpha_3^2 u_3(\omega)u_3(\phi_2(\omega)) = \frac{1}{4}.$$

Since we have $\alpha_2 + \alpha_3 = \frac{1}{2}$, it follows that

$$u_2(\omega)u_2(\phi_2(\omega)) = u_2(\omega)u_3(\phi_2(\omega)) = u_3(\omega)u_2(\phi_2(\omega)) = u_3(\omega)u_3(\phi_2(\omega)) = 1.$$

This implies that $u_2(\omega) = u_3(\omega)$ and $u_2(\phi_2(\omega)) = u_3(\phi_2(\omega))$. □

Lemma 3.2.9. *If $\omega \in D_i$, $i = 1, 2, 3$ then $\alpha_1 = \alpha_2 = \alpha_3 = 1/3$.*

Proof. Let $\omega \in D_1$, that is, $\omega = \phi_1(\omega) \neq \phi_2(\omega) \neq \phi_3(\omega) \neq \omega$. Equation $(**)$ reduces to

$$\alpha_1 u_1(\omega)[\alpha_1 u_1(\omega)f(\omega) + \alpha_2 u_2(\omega)f(\phi_2(\omega)) + \alpha_3 u_3(\omega)f(\phi_3(\omega))] +$$
$$\alpha_2 u_2(\omega)[\alpha_1 u_1(\phi_2(\omega))f(\phi_1 \circ \phi_2(\omega)) + \alpha_2 u_2(\phi_2(\omega))f(\phi_2^2(\omega)) + \alpha_3 u_3(\phi_2(\omega))f(\phi_3 \circ \phi_2(\omega))] +$$
$$\alpha_3 u_3(\omega)[\alpha_1 u_1(\phi_3(\omega))f(\phi_1 \circ \phi_3(\omega)) + \alpha_2 u_2(\phi_3(\omega))f(\phi_2 \circ \phi_3(\omega)) + \alpha_3 u_3(\phi_3(\omega))f(\phi_3^2(\omega))] =$$
$$\alpha_1 u_1(\omega)f(\omega) + \alpha_2 u_2(\omega)f(\phi_2(\omega)) + \alpha_3 u_3(\omega)f(\phi_3(\omega)). \tag{3.2.16}$$

We divide the proof into three steps.

Step I. We claim that if $\omega = \phi_2 \circ \phi_i(\omega)$, $i = 2$ or 3, then $\omega = \phi_3 \circ \phi_j(\omega)$, $j = 2$ or 3.

Step II. We partition the set D_1 into six disjoint sets and for each of these sets we show that $\alpha_1 = \alpha_2 = \alpha_3 = 1/3$.

Step III. We obtain conditions on $u_i(\omega)$ and $u_i(\phi_j(\omega))$, $i = 1, 2, 3$ and $j = 2, 3$, for each partitioned set.

Proof of Step I

Let $f \in C(\Omega)$ satisfies $f(\omega) = 1,\ f(\phi_2(\omega)) = f(\phi_3(\omega)) = f(\phi_1 \circ \phi_2(\omega)) = f(\phi_1 \circ \phi_3(\omega)) = 0$. Then Equation (3.2.16) becomes

$$\alpha_1^2 u_1^2(\omega) + \alpha_2 u_2(\omega)[\alpha_2 u_2(\phi_2(\omega))f(\phi_2^2(\omega)) + \alpha_3 u_3(\phi_2(\omega))(\phi_3 \circ \phi_2(\omega))] +$$
$$\alpha_3 u_3(\omega)[\alpha_2 u_2(\phi_3(\omega))f(\phi_2 \circ \phi_3(\omega)) \ + \alpha_3 u_3(\phi_3(\omega))f(\phi_3^2(\omega))] = \alpha_1 u_1(\omega). \tag{3.2.17}$$

If $\phi_2^2(\omega)$, $\phi_3 \circ \phi_2(\omega)$, $\phi_2 \circ \phi_3(\omega)$ and $\phi_3^2(\omega)$ are all different from ω, by choosing our function f to take value 0 at all these points we will have $\alpha_1^2 u_1^2(\omega) = \alpha_1 u_1(\omega)$ and hence $\alpha_1 = 1$. Thus, not all these points are different from ω.

Claim: If $\omega = \phi_2 \circ \phi_i(\omega)$, $i = 2$ or 3, then $\omega = \phi_3 \circ \phi_j(\omega)$, $j = 2$ or 3.

To see the proof of the claim, let $\omega = \phi_2 \circ \phi_i(\omega)$, $i = 2$ or 3, then in Equation (3.2.17), $f(\phi_2 \circ \phi_j(\omega)) = 0$, $j = 2$ or 3 and $j \neq i$. Suppose to the contrary that $\omega \neq \phi_3 \circ \phi_k(\omega)$ for $k = 2, 3$ then by choosing our f to be 0 at these points we get from Equation (3.2.17)

$$\alpha_1^2 u_1^2(\omega) + \alpha_2 \alpha_i u_i(\omega) u_2(\phi_i(\omega)) = \alpha_1 u_1(\omega). \tag{3.2.18}$$

This implies that

$$\alpha_1 \leq \alpha_1^2 + \alpha_2 \alpha_i. \tag{3.2.19}$$

We now choose a function $f \in C(\Omega)$ such that $f(\phi_2(\omega)) = 1$ and $f(\omega) = f(\phi_3(\omega)) = f(\phi_2^2(\omega)) = f(\phi_2 \circ \phi_3(\omega)) = 0$. Then Equation (3.2.16) is reduced to

$$\alpha_1 \alpha_2 u_1(\omega) u_2(\omega) + \alpha_2 u_2(\omega)[\alpha_1 u_1(\phi_2(\omega)) f(\phi_1 \circ \phi_2(\omega)) + \alpha_3 u_3(\phi_2(\omega)) f(\phi_3 \circ \phi_2(\omega))] +$$

$$\alpha_3 u_3(\omega)[\alpha_1 u_1(\phi_3(\omega)) f(\phi_1 \circ \phi_3(\omega)) + \alpha_3 u_3(\phi_3(\omega)) f(\phi_3^2(\omega))] = \alpha_2 u_2(\omega). \tag{3.2.20}$$

Again, if all $\phi_1 \circ \phi_2(\omega)$, $\phi_3 \circ \phi_2(\omega)$, $\phi_1 \circ \phi_3(\omega)$ and $\phi_3^2(\omega)$ are different from $\phi_2(\omega)$, by choosing f initially to take value 0 at all these points we could have $\alpha_1 = 1$. Suppose $\phi_2(\omega) = \phi_1 \circ \phi_{i_1}(\omega)$ where $i_1 = 2$ or 3. Then we could choose f in Equation (3.2.20) such that $f(\phi_1 \circ \phi_{i_2}(\omega)) = 0$, $i_2 = 2$ or 3 and $i_2 \neq i_1$. If $\phi_2(\omega) \neq \phi_3 \circ \phi_{i_3}(\omega)$, $i_3 = 2, 3$. Then by the same argument we get from Equation (3.2.20)

$$\alpha_1 \alpha_2 u_1(\omega) u_2(\omega) + \alpha_1 \alpha_{i_1} u_{i_1}(\omega) u_1(\phi_{i_1}(\omega)) = \alpha_2 u_2(\omega). \tag{3.2.21}$$

This implies that

$$\alpha_2 \leq \alpha_1(\alpha_2 + \alpha_{i_1}). \tag{3.2.22}$$

If $i = i_1$, then adding the Inequalities (3.2.19) and (3.2.22) we get $\alpha_1 + \alpha_i \geq 1$ - A contradiction.

If $i = 2$ and $i_1 = 3$, then Inequality (3.2.22) becomes $\alpha_2 \leq \alpha_1(\alpha_2 + \alpha_3) = \alpha_1(1 - \alpha_1)$. Adding this and Inequality (3.2.19) we get $\alpha_1 + \alpha_2 \leq \alpha_1 + \alpha_2^2$ or $\alpha_2 \geq 1$ - A contradiction again.

If $i = 3$ and $i_1 = 2$, then from Inequality (3.2.22) we have $\alpha_1 \geq 1/2$. Since $\alpha_k \leq 1/2$ for all $k = 1, 2, 3$, we conclude that $\alpha_1 = 1/2$. Now, Inequality (3.2.19) is reduced to $1/4 \leq \alpha_2\alpha_3 \leq \alpha_2/2$ or $\alpha_2 \geq 1/2$. This implies that $\alpha_2 = 1/2$ - A contradiction.

Therefore, we get $\phi_2(\omega) = \phi_3 \circ \phi_{i_4}(\omega)$, $i_4 = 2$ or 3. Choosing a function f such that $f(\omega) = f(\phi_2(\omega)) = f(\phi_3(\omega)) = 0$ in Equation (3.2.16) we will be left with three points, that is, $\phi_1 \circ \phi_{i_5}(\omega)$ $(i_5 \neq i_1)$, $\phi_2 \circ \phi_{i_6}(\omega)$ $(i_6 \neq i)$ and $\phi_3 \circ \phi_{i_7}(\omega)$ $(i_7 \neq i_4)$, $i_5, i_6, i_7 = 2$ or 3, and we have 0 on the right hand side. It is also clear that $\phi_3 \circ \phi_{i_7}(\omega)$ is not equal to any of ω (because of our assumption), $\phi_2(\omega)$ or $\phi_3(\omega)$. So, it has to be equal to at least one of $\phi_1 \circ \phi_{i_5}(\omega)$ or $\phi_2 \circ \phi_{i_6}(\omega)$. But in all these cases we can choose f large enough to get a contradiction.

Proof of Step II

Choosing a function $f \in C(\Omega)$ such that $f(\phi_2(\omega)) = 1,\ f(\omega) = f(\phi_3(\omega)) = f(\phi_2^2(\omega)) = f(\phi_2 \circ \phi_3(\omega)) = 0$ and then a function f such that $f(\phi_3(\omega)) = 1,\ f(\omega) = f(\phi_2(\omega)) = f(\phi_3^2(\omega)) = f(\phi_3 \circ \phi_2(\omega)) = 0$ in Equation (3.2.16), we will get the following two equations.

$$\alpha_1\alpha_2 u_1(\omega)u_2(\omega) + \alpha_2 u_2(\omega)[\alpha_1 u_1(\phi_2(\omega))f(\phi_1 \circ \phi_2(\omega)) + \alpha_3 u_3(\phi_2(\omega))f(\phi_3 \circ \phi_2(\omega))] + \alpha_3 u_3(\omega)[\alpha_1 u_1(\phi_3(\omega))f(\phi_1 \circ \phi_3(\omega)) + \alpha_3 u_3(\phi_3(\omega))f(\phi_3^2(\omega))] = \alpha_2 u_2(\omega). \tag{3.2.23}$$

$$\alpha_1\alpha_3 u_1(\omega)u_3(\omega) + \alpha_2 u_2(\omega)[\alpha_1 u_1(\phi_2(\omega))f(\phi_1 \circ \phi_2(\omega)) + \alpha_2 u_2(\phi_2(\omega))f(\phi_2^2(\omega))] + \alpha_3 u_3(\omega)[\alpha_1 u_1(\phi_3(\omega))f(\phi_1 \circ \phi_3(\omega)) + \alpha_2 u_2(\phi_3(\omega))f(\phi_2 \circ \phi_3(\omega))] = \alpha_3 u_3(\omega). \tag{3.2.24}$$

From the above claim we have the following disjoint and exhaustive cases which may occur.

$D_{11} = \{\omega \in D_1 :\ \omega = \phi_2^2(\omega) = \phi_3 \circ \phi_2(\omega),\ \phi_2(\omega) = \phi_3^2(\omega) = \phi_1 \circ \phi_2(\omega),\ \phi_3(\omega) = \phi_1 \circ \phi_3(\omega) = \phi_2 \circ \phi_3(\omega)\}$,

$D_{12} = \{\omega \in D_1 :\ \omega = \phi_2^2(\omega) = \phi_3 \circ \phi_2(\omega),\ \phi_2(\omega) = \phi_3^2(\omega) = \phi_1 \circ \phi_3(\omega),\ \phi_3(\omega) = \phi_1 \circ \phi_2(\omega) = \phi_2 \circ \phi_3(\omega)\}$,

$D_{13} = \{\omega \in D_1 : \ \omega = \phi_2 \circ \phi_3(\omega) = \phi_3 \circ \phi_2(\omega), \ \phi_2(\omega) = \phi_3^2(\omega) = \phi_1 \circ \phi_2(\omega), \ \phi_3(\omega) = \phi_1 \circ \phi_3(\omega) = \phi_2^2(\omega)\}$,

$D_{14} = \{\omega \in D_1 : \ \omega = \phi_2 \circ \phi_3(\omega) = \phi_3 \circ \phi_2(\omega), \ \phi_2(\omega) = \phi_3^2(\omega) = \phi_1 \circ \phi_3(\omega), \ \phi_3(\omega) = \phi_1 \circ \phi_2(\omega) = \phi_2^2(\omega)\}$,

$D_{15} = \{\omega \in D_1 : \ \omega = \phi_2^2(\omega) = \phi_3^2(\omega), \ \phi_2(\omega) = \phi_1 \circ \phi_2(\omega) = \phi_3 \circ \phi_2(\omega), \ \phi_3(\omega) = \phi_1 \circ \phi_3(\omega) = \phi_2 \circ \phi_3(\omega)\}$,

$D_{16} = \{\omega \in D_1 : \ \omega = \phi_2^2(\omega) = \phi_3^2(\omega), \ \phi_2(\omega) = \phi_1 \circ \phi_3(\omega) = \phi_3 \circ \phi_2(\omega), \ \phi_3(\omega) = \phi_1 \circ \phi_2(\omega) = \phi_2 \circ \phi_3(\omega)\}$.

Now for any $\omega \in D_{11}$, Equation (3.2.16) is reduced to

$$\begin{aligned}
&\{\alpha_1^2 u_1^2(\omega) + \alpha_2 u_2(\omega)[\alpha_2 u_2(\phi_2(\omega)) + \alpha_3 u_3(\phi_2(\omega))]\} f(\omega) + \\
&\{\alpha_1\alpha_2 u_1(\omega)u_2(\omega) + \alpha_1\alpha_2 u_1(\phi_2(\omega))u_2(\omega) + \alpha_3^2 u_3(\omega)u_3(\phi_3(\omega))\} f(\phi_2(\omega)) + \\
&\{\alpha_1\alpha_3 u_1(\omega)u_3(\omega) + \alpha_3 u_3(\omega)[\alpha_1 u_1(\phi_3(\omega)) + \alpha_2 u_2(\phi_3(\omega))]\} f(\phi_3(\omega)) \\
&= \alpha_1 u_1(\omega) f(\omega) + \alpha_2 u_2(\omega) f(\phi_2(\omega)) + \alpha_3 u_3(\omega) f(\phi_3(\omega)). \qquad (3.2.25)
\end{aligned}$$

Since $\omega \neq \phi_2(\omega) \neq \phi_3(\omega) \neq \omega$, choosing appropriate functions we get

$$\alpha_1 \leq \alpha_1^2 + \alpha_2(\alpha_2 + \alpha_3), \ \alpha_2 \leq 2\alpha_1\alpha_2 + \alpha_3^2 \text{ and } 1 \leq 2\alpha_1 + \alpha_2. \qquad (3.2.26)$$

For $\omega \in D_{12}$, we have

$$\begin{aligned}
&\{\alpha_1^2 u_1^2(\omega) + \alpha_2 u_2(\omega)[\alpha_2 u_2(\phi_2(\omega)) + \alpha_3 u_3(\phi_2(\omega))]\} f(\omega) + \\
&\{\alpha_1\alpha_2 u_1(\omega)u_2(\omega) + \alpha_3 u_3(\omega)[\alpha_1 u_1(\phi_3(\omega)) + \alpha_3 u_3(\phi_3(\omega))]\} f(\phi_2(\omega)) + \\
&\{\alpha_1\alpha_3 u_1(\omega)u_3(\omega) + \alpha_1\alpha_2 u_2(\omega)u_1(\phi_2(\omega)) + \alpha_2\alpha_3 u_3(\omega)u_2(\phi_3(\omega))\} f(\phi_3(\omega)) \\
&= \alpha_1 u_1(\omega) f(\omega) + \alpha_2 u_2(\omega) f(\phi_2(\omega)) + \alpha_3 u_3(\omega) f(\phi_3(\omega)). \qquad (3.2.27)
\end{aligned}$$

This implies that

$$\begin{aligned}
&\alpha_1 \leq \alpha_1^2 + \alpha_2(\alpha_2 + \alpha_3), \ \alpha_2 \leq \alpha_1\alpha_2 + \alpha_3(\alpha_1 + \alpha_3) \text{ and} \\
&\alpha_3 \leq \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1. \qquad (3.2.28)
\end{aligned}$$

For $\omega \in D_{13}$, we have

$$\{\alpha_1^2 u_1^2(\omega) + \alpha_2\alpha_3[u_2(\omega)u_3(\phi_2(\omega)) + u_3(\omega)u_2(\phi_3(\omega))]\}f(\omega) +$$

$$\{\alpha_1\alpha_2 u_1(\omega)u_2(\omega) + \alpha_1\alpha_2 u_2(\omega)u_1(\phi_2(\omega)) + \alpha_3^2 u_3(\omega)u_3(\phi_3(\omega))\}f(\phi_2(\omega)) +$$

$$\{\alpha_1\alpha_3 u_1(\omega)u_3(\omega) + \alpha_2^2 u_2(\omega)u_2(\phi_2(\omega)) + \alpha_1\alpha_3 u_3(\omega)u_1(\phi_3(\omega))\}f(\phi_3(\omega))$$

$$= \alpha_1 u_1(\omega)f(\omega) + \alpha_2 u_2(\omega)f(\phi_2(\omega)) + \alpha_3 u_3(\omega)f(\phi_3(\omega)). \tag{3.2.29}$$

This implies that

$$\alpha_1 \leq \alpha_1^2 + 2\alpha_2\alpha_3,\ \alpha_2 \leq 2\alpha_1\alpha_2 + \alpha_3^2 \text{ and } \alpha_3 \leq 2\alpha_1\alpha_3 + \alpha_2^2. \tag{3.2.30}$$

For $\omega \in D_{14}$, we have

$$\{\alpha_1^2 u_1^2(\omega) + \alpha_2\alpha_3[u_2(\omega)u_3(\phi_2(\omega)) + u_3(\omega)u_2(\phi_3(\omega))]\}f(\omega) +$$

$$\{\alpha_1\alpha_2 u_1(\omega)u_2(\omega) + \alpha_3 u_3(\omega)[\alpha_1 u_1(\phi_3(\omega)) + \alpha_3 u_3(\phi_3(\omega))]\}f(\phi_2(\omega)) +$$

$$\{\alpha_1\alpha_3 u_1(\omega)u_3(\omega) + \alpha_2 u_2(\omega)[\alpha_1 u_1(\phi_2(\omega)) + \alpha_2 u_2(\phi_2(\omega))]\}f(\phi_3(\omega))$$

$$= \alpha_1 u_1(\omega)f(\omega) + \alpha_2 u_2(\omega)f(\phi_2(\omega)) + \alpha_3 u_3(\omega)f(\phi_3(\omega)). \tag{3.2.31}$$

This implies that

$$\alpha_1 \leq \alpha_1^2 + 2\alpha_2\alpha_3,\ \alpha_2 \leq \alpha_1\alpha_2 + \alpha_3(\alpha_1 + \alpha_3) \text{ and}$$

$$\alpha_3 \leq \alpha_1\alpha_3 + \alpha_2(\alpha_1 + \alpha_2). \tag{3.2.32}$$

For $\omega \in D_{15}$, we have

$$\{\alpha_1^2 u_1^2(\omega) + \alpha_2^2 u_2(\omega)u_2(\phi_2(\omega)) + \alpha_3^2 u_3(\omega)u_3(\phi_3(\omega))\}f(\omega) +$$

$$\{\alpha_1\alpha_2 u_1(\omega)u_2(\omega) + \alpha_2 u_2(\omega)[\alpha_1 u_1(\phi_2(\omega)) + \alpha_3 u_3(\phi_2(\omega))]\}f(\phi_2(\omega)) +$$

$$\{\alpha_1\alpha_3 u_1(\omega)u_3(\omega) + \alpha_3 u_3(\omega)[\alpha_1 u_1(\phi_3(\omega)) + \alpha_2 u_2(\phi_3(\omega))]\}f(\phi_3(\omega))$$

$$= \alpha_1 u_1(\omega)f(\omega) + \alpha_2 u_2(\omega)f(\phi_2(\omega)) + \alpha_3 u_3(\omega)f(\phi_3(\omega)). \tag{3.2.33}$$

This implies that

$$\alpha_1 \leq \alpha_1^2 + \alpha_2^2 + \alpha_3^2,\ 1 \leq 2\alpha_1 + \alpha_3 \text{ and } 1 \leq 2\alpha_1 + \alpha_2. \tag{3.2.34}$$

For $\omega \in D_{16}$, we have

$$\begin{aligned} &\{\alpha_1^2 u_1^2(\omega) + \alpha_2^2 u_2(\omega)u_2(\phi_2(\omega)) + \alpha_3^2 u_3(\omega)u_3(\phi_3(\omega))\}f(\omega) + \\ &\{\alpha_1\alpha_2 u_1(\omega)u_2(\omega) + \alpha_2\alpha_3 u_2(\omega)u_3(\phi_2(\omega)) + \alpha_1\alpha_3 u_3(\omega)u_1(\phi_3(\omega))\}f(\phi_2(\omega)) + \\ &\{\alpha_1\alpha_3 u_1(\omega)u_3(\omega) + \alpha_1\alpha_2 u_2(\omega)u_1(\phi_2(\omega)) + \alpha_2\alpha_3 u_3(\omega)u_2(\phi_3(\omega))\}f(\phi_3(\omega)) \\ &= \alpha_1 u_1(\omega)f(\omega) + \alpha_2 u_2(\omega)f(\phi_2(\omega)) + \alpha_3 u_3(\omega)f(\phi_3(\omega)). \end{aligned} \tag{3.2.35}$$

This implies that

$$\begin{aligned} &\alpha_1 \leq \alpha_1^2 + \alpha_2^2 + \alpha_3^2,\ \alpha_2 \leq \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1 \text{ and} \\ &\alpha_3 \leq \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1. \end{aligned} \tag{3.2.36}$$

To summarize, we have the following equations.

$$\alpha_1 \leq \alpha_1^2 + \alpha_2(\alpha_2 + \alpha_3),\ \alpha_2 \leq 2\alpha_1\alpha_2 + \alpha_3^2 \text{ and } 1 \leq 2\alpha_1 + \alpha_2. \tag{3.2.37}$$

$$\alpha_1 \leq \alpha_1^2 + \alpha_2(\alpha_2 + \alpha_3),\ \alpha_2 \leq \alpha_1\alpha_2 + \alpha_3(\alpha_1 + \alpha_3) \text{ and } \alpha_3 \leq \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1. \tag{3.2.38}$$

$$\alpha_1 \leq \alpha_1^2 + 2\alpha_2\alpha_3,\ \alpha_2 \leq 2\alpha_1\alpha_2 + \alpha_3^2 \text{ and } \alpha_3 \leq 2\alpha_1\alpha_3 + \alpha_2^2. \tag{3.2.39}$$

$$\alpha_1 \leq \alpha_1^2 + 2\alpha_2\alpha_3,\ \alpha_2 \leq \alpha_1\alpha_2 + \alpha_3(\alpha_1 + \alpha_3) \text{ and } \alpha_3 \leq \alpha_1\alpha_3 + \alpha_2(\alpha_1 + \alpha_2). \tag{3.2.40}$$

$$\alpha_1 \leq \alpha_1^2 + \alpha_2^2 + \alpha_3^2,\ 1 \leq 2\alpha_1 + \alpha_3 \text{ and } 1 \leq 2\alpha_1 + \alpha_2. \tag{3.2.41}$$

$$\alpha_1 \leq \alpha_1^2 + \alpha_2^2 + \alpha_3^2,\ \alpha_2 \leq \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1 \text{ and } \alpha_3 \leq \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1. \tag{3.2.42}$$

In the above six equations, it is easy to observe that $\alpha_i = 1/3,\ i = 1, 2, 3$ is the only solution.

Proof of Step III

In this step we will find conditions on $u_i(\omega)$ and $u_i(\phi_j(\omega))$ for $i = 1, 2, 3$ and $j = 2, 3$. We substitute $\alpha_i = 1/3$ in Equations (3.2.25), (3.2.27), (3.2.29), (3.2.31), (3.2.33) and (3.2.35) and we choose three sets of functions for each equation. Firstly, a function $f \in C(\Omega)$ such that $f(\omega) = 1,\ f(\phi_2(\omega)) = f(\phi_3(\omega)) = 0$. Then a function $f \in C(\Omega)$ such that $f(\phi_2(\omega)) = 1,\ f(\omega) = f(\phi_3(\omega)) = 0$ and finally a function $f \in C(\Omega)$ such that $f(\phi_3(\omega)) = 1,\ f(\omega) = f(\phi_2(\omega)) = 0$. Moreover, by observing that $u_i(\omega)$ and $u_i(\phi_j(\omega))$ lie on the unit circle and all the points on the circle are extreme points we get the following conditions on $u_i(\omega)$ and $u_i(\phi_j(\omega))$ where $i = 1, 2, 3$ and $j = 2, 3$.

For $\omega \in D_{11}$ we get

$$u_1(\omega) = u_2(\omega)u_2(\phi_2(\omega)) = u_2(\omega)u_3(\phi_2(\omega)) = 1,\ u_1(\phi_2(\omega)) = 1,$$

$$u_3(\omega)u_3(\phi_3(\omega)) = u_2(\omega) \text{ and } u_1(\phi_3(\omega)) = u_2(\phi_3(\omega)) = 1.$$

For $\omega \in D_{12}$ we get

$$u_1(\omega) = u_2(\omega)u_2(\phi_2(\omega)) = u_2(\omega)u_3(\phi_2(\omega)) = 1,\ u_3(\omega)u_1(\phi_3(\omega))$$

$$= u_3(\omega)u_3(\phi_3(\omega)) = u_2(\omega) \text{ and } u_2(\omega)u_1(\phi_2(\omega)) = u_3(\omega),\ u_2(\phi_3(\omega)) = 1$$

For $\omega \in D_{13}$ we get

$$u_1(\omega) = u_2(\omega)u_3(\phi_2(\omega)) = u_3(\omega)u_2(\phi_3(\omega)) = 1,\ u_1(\phi_2(\omega)) = 1,$$

$$u_3(\omega)u_3(\phi_3(\omega)) = u_2(\omega) \text{ and } u_2(\omega)u_2(\phi_2(\omega)) = u_3(\omega),\ u_1(\phi_3(\omega)) = 1.$$

For $\omega \in D_{14}$ we get

$$u_1(\omega) = u_2(\omega)u_3(\phi_2(\omega)) = u_3(\omega)u_2(\phi_3(\omega)) = 1,\ u_3(\omega)u_1(\phi_3(\omega)) =$$

$$u_3(\omega)u_3(\phi_3(\omega)) = u_2(\omega) \text{ and } u_2(\omega)u_1(\phi_2(\omega)) = u_2(\omega)u_2(\phi_2(\omega)) = u_3(\omega).$$

For $\omega \in D_{15}$ we get

$$u_1(\omega) = u_2(\omega)u_2(\phi_2(\omega)) = u_3(\omega)u_3(\phi_3(\omega)) = 1,$$

$$u_1(\phi_2(\omega)) = u_3(\phi_2(\omega)) = 1 \text{ and } u_1(\phi_3(\omega)) = u_2(\phi_3(\omega)) = 1.$$

For $\omega \in D_{16}$ we get

$$u_1(\omega) = u_2(\omega)u_2(\phi_2(\omega)) = u_3(\omega)u_3(\phi_3(\omega)) = 1, \ u_3(\phi_2(\omega)) = 1,$$

$$u_3(\omega)u_1(\phi_3(\omega)) = u_2(\omega) \text{ and } u_2(\omega)u_1(\phi_2(\omega)) = u_3(\omega), \ u_2(\phi_3(\omega)) = 1.$$

Thus, the proof of the Lemma is complete. □

We will need one more lemma to complete the proof of Theorem 3.1.2.

Lemma 3.2.10. *With the assumption in Theorem 3.1.2, one and only one of the following conditions is possible: (In all the cases $i, j, k = 1, 2, 3$)*

(i) $\Omega = B_i$.

(ii) $\Omega = A \cup B_i$.

(iii) $\Omega = A \cup B_i \cup C_i$.

(iv) $\Omega = C_i$.

(v) $\Omega = A \cup C_i$.

(vi) $\Omega = D_{ij}$.

(vii) $\Omega = A \cup D_{ij}$.

(viii) $\Omega = A \cup D_{ij} \cup D_{kl}$, $l = 1, \ldots, 6$.

(ix) $\Omega = A \cup D_{1i} \cup D_{2j} \cup D_{3k}$.

Proof. Suppose that $\Omega = A \cup B_1 \cup B_2 \cup B_3$. Let ω be a limit point of B_i, $i = 1, 2, 3$. Then there exists a net $\omega_\alpha \in B_i$ such that $\omega_\alpha \to \omega$. Since $\omega_\alpha \in B_i$, we have $\omega_\alpha = \phi_j(\omega_\alpha) = \phi_k(\omega_\alpha) \neq \phi_i(\omega_\alpha)$. This implies that $\omega = \phi_j(\omega) = \phi_k(\omega)$ and hence $\omega \in A \cup B_i$.

We now consider the following cases:

(a) If all B_i's are closed, then by the connectedness of Ω and observing that A is closed, we have $\Omega = A$, $\Omega = B_1$, $\Omega = B_2$ or $\Omega = B_3$. Since $\Omega \neq A$, we conclude that $\Omega = B_1$, $\Omega = B_2$ or $\Omega = B_3$. Thus, (i) is proved.

(b) If only one of the B_i's, say B_j, is closed. Then as we have shown earlier, any limit point of B_j belongs to $A \cup B_j$. Thus, $A \cup B_i \cup B_k$ is closed. Hence, by the connectedness of Ω, either $\Omega = B_j$ or $\Omega = A \cup B_i \cup B_k$.

Suppose that B_3 is closed and $\Omega = A \cup B_1 \cup B_2$. Rest of the cases are exactly similar. Since B_1 is not closed, there exists a net $\omega_\alpha \in B_1$ such that $\omega_\alpha \to \omega$ and $\omega \in A$. This implies that $\phi_1(\omega) = \phi_2(\omega) = \phi_3(\omega)$.

If $\omega \in A_1$, then $u_1(\omega) = u_2(\omega) = u_3(\omega) = 1$ and $\phi_1(\omega) = \omega$. Since $\omega_\alpha \in B_1$, we have from Equation (3.2.7)

$$\alpha_1^2 u_1(\omega_\alpha) u_1(\phi_1(\omega_\alpha)) + [\alpha_2 u_2(\omega_\alpha) + \alpha_3 u_3(\omega_\alpha)]^2 = \alpha_2 u_2(\omega_\alpha) + \alpha_3 u_3(\omega_\alpha). \tag{3.2.43}$$

Noting that each of u_1, u_2 and u_3 are continuous and taking the limit both sides we get

$$\alpha_1^2 u_1(\omega) u_1(\phi_1(\omega)) + [\alpha_2 u_2(\omega) + \alpha_3 u_3(\omega)]^2 = \alpha_2 u_2(\omega) + \alpha_3 u_3(\omega). \tag{3.2.44}$$

Since $u_i(\omega) = 1$, $i = 1, 2, 3$ and $\phi_1(\omega) = \omega$, we have $\alpha_1^2 + [\alpha_2 + \alpha_3]^2 = \alpha_2 + \alpha_3$ or $\alpha_1 = 1/2$.

If $\omega \in A_2$, then $\alpha_1 u_1(\omega) + \alpha_2 u_2(\omega) + \alpha_3 u_3(\omega) = 0$. Since $\omega_\alpha \in B_1$, Equation (3.2.9) shows that

$$\alpha_2 u_2(\omega_\alpha) + \alpha_3 u_3(\omega_\alpha) + \alpha_2 u_2(\phi_1(\omega_\alpha)) + \alpha_3 u_3(\phi_1(\omega_\alpha)) = 1. \tag{3.2.45}$$

Taking limits we get

$$\alpha_2 u_2(\omega) + \alpha_3 u_3(\omega) + \alpha_2 u_2(\phi_1(\omega)) + \alpha_3 u_3(\phi_1(\omega)) = 1 \tag{3.2.46}$$

or $-\alpha_1 u_1(\omega) - \alpha_1 u_1(\omega) = 1$. This implies that $\alpha_1 = 1/2$.

Similar argument for B_2 will give us $\alpha_2 = 1/2$ - A contradiction.

Thus, $\Omega \neq A \cup B_1 \cup B_2$.

(c) If two of the B_i's, say B_j and B_k, are closed, then we will have $\Omega = B_j$, $\Omega = B_k$ or $\Omega = A \cup B_i$. Thus, (ii) is proved.

Suppose $\Omega = A \cup B_i$. Then as B_i is not closed, there exists a net $\omega_\alpha \in B_i$ such that $\omega_\alpha \to \omega$ and $\omega \in A$. Proceeding as above we conclude that $\alpha_i = 1/2$ and from Equation (3.2.9) and (3.2.7) we will get $u_j(\omega) = u_k(\omega) = u_j(\phi_i(\omega)) = u_k(\phi_i(\omega)) = u_i(\omega)u_i(\phi_i(\omega)) = 1$ for all $\omega \in B_i$.

(d) Suppose that no B_i is closed, then $\Omega = A \cup B_1 \cup B_2 \cup B_3$. Proceeding in the same way as in (b), we can see that this case is also not possible.

From Lemma 3.2.8 one can see that none of C_1, C_2 or C_3 can occur together. Suppose that $\Omega = A \cup B_i \cup B_j \cup B_k \cup C_i$. Lemma 3.2.8 also implies that $\alpha_i = 1/2$.

We claim that B_j, B_k and $A \cup B_i \cup C_i$ are closed. To see the claim, suppose to the contrary that B_j is not closed. Then there exists a net $\omega_\alpha \in B_j$ such that $\omega_\alpha \to \omega$ and $\omega \in A$. An argument similar to case (b) above will give us $\alpha_j = 1/2$, which is a contradiction since $\alpha_i = 1/2$.

Similarly, one can show that B_k is closed.

Now, let ω be a limit point C_i. Then there exists a net $\omega_\alpha \in C_i$ such that $\omega_\alpha \to \omega$. Since $\omega_\alpha \in C_i$, $\omega = \phi_i(\omega_\alpha) \neq \phi_j(\omega_\alpha) = \phi_k(\omega_\alpha)$. This implies that $\omega = \phi_i(\omega)$, $\phi_j(\omega) = \phi_k(\omega)$ and hence $\omega \in A \cup C_i$. We have seen earlier that limits points of B_i belong to $A \cup B_i$. Therefore, $A \cup B_i \cup C_i$ is also closed. From connectedness of Ω we conclude that $\Omega = B_j$, $\Omega = B_k$ or $A \cup B_i \cup C_i$.

Let $\Omega = A \cup B_i \cup C_i$. If both B_i and C_i are closed, then $\Omega = B_i$ or $\Omega = C_i$. If only B_i is closed, then $\Omega = B_i$ or $\Omega = A \cup C_i$. If only C_i is closed, then $\Omega = C_i$ or $\Omega = A \cup B_i$. If neither B_i nor C_i is closed, then $\Omega = A \cup B_i \cup C_i$.

Suppose $\Omega = A \cup B_i \cup C_i$, then as $\alpha_i = 1/2$, from Equation (3.2.9) and (3.2.7) we will get $u_j(\omega) = u_k(\omega) = u_j(\phi_i(\omega)) = u_k(\phi_i(\omega)) = u_i(\omega)u_i(\phi_i(\omega)) = 1$ for all $\omega \in B_i$.

This proves assertions $(iii) - (v)$.

It is clear from Lemma 3.2.8 and 3.2.9 that for $i = 1, 2, 3$ C_i cannot occur with D_i. We also observe that any limit point of D_i belongs to $A \cup D_i$. Moreover, for fixed $i = 1, 2, 3$ no two or more D_{ij}, $j = 1, \ldots, 6$ can occur simultaneously.

Suppose that $\Omega = A \cup B_i \cup D_{1j} \cup D_{2k} \cup D_{3l}$, where $i = 1, 2, 3$ and $j, k, l = 1, 2, \ldots, 6$. Lemma 3.2.9 shows that $\alpha_1 = \alpha_2 = \alpha_3 = 1/3$.

Suppose that neither B_i nor any one of D_{mn}'s is closed, where $m = 1, 2, 3$ and $n = j, k, l$. Then we will get $\alpha_i = 1/2$ by an argument similar to case (b) above and this will lead us to a contradiction. So, no B_i can occur with any of the D_{mn}'s and hence either B_i or all of D_{mn}'s must be closed.

If B_i is not closed, then we will have $\Omega = D_{mn}$ for some m and n. Hence, assertion (vi) is proved.

If B_i is closed, then either $\Omega = B_i$ or $\Omega = A \cup D_{1j} \cup D_{2k} \cup D_{3l}$. If $\Omega = A \cup D_{1j} \cup D_{2k} \cup D_{3l}$ and some D_{mn}'s are closed, then by arguing in a similar way we will get cases $(vii) - (ix)$.

The proof of the Lemma is complete. □

Completion of the proof of Theorem 3.1.2

In any of the cases $(i) - (v)$ in Lemma 3.2.10, we have seen that

$u_j(\omega) = u_k(\omega) = u_j(\phi_i(\omega)) = u_k(\phi_i(\omega)) = u_i(\omega)u_i(\phi_i(\omega)) = 1$ for any $\omega \in B_i$ and

$u_i(\omega) = u_i(\phi_j(\omega)) = 1$, $u_j(\omega) = u_k(\omega)$, $u_j(\phi_j(\omega)) = u_k(\phi_j(\omega))$ and $u_j(\omega)u_j(\phi_j(\omega)) = 1$ for any $\omega \in C_i$. Moreover, $\alpha_i = 1/2$.

Therefore, we have $T_j f(\omega) = T_k f(\omega)$ for all $f \in C(\Omega)$, $\omega \in B_i \cup C_i$. Thus, we have $P = \frac{T_i + T_j}{2}$.

Therefore, the proof of Theorem 3.1.2 (a) is complete.

It remains to consider the case when $\Omega = A \cup D_{1i} \cup D_{2j} \cup D_{3k}$. We further assume that $i, k \leq 4,\ j \geq 5$. The remaining cases for the conditions $(vi) - (viii)$ are similar.

Our aim is to show that there exists a surjective isometry T on $C(\Omega)$ such that $T^3 = I$ and $P = \frac{I+T+T^2}{3}$. Since $P = \frac{1}{3}(T_1 + T_2 + T_3)$ is a projection, we have

$$P = \frac{1}{9}(T_1^2 + T_2^2 + T_3^2 + T_1T_2 + T_2T_1 + T_1T_3 + T_3T_1 + T_2T_3 + T_3T_2).$$

Using the conditions obtained earlier on $u_i(\omega)$'s and $u_i(\phi_j(\omega))$ we see that for any $\omega \in D_{11}$;

$$T_1^2 f(\omega) = T_2^2 f(\omega) = f(\omega),\ T_3^2 f(\omega) = T_2 f(\omega),\ T_1T_2 f(\omega) = T_2T_1 f(\omega) = T_2 f(\omega),$$

$$T_1T_3 f(\omega) = T_3T_1 f(\omega) = T_3T_2 f(\omega) = T_3 f(\omega),\ T_2T_3 f(\omega) = f(\omega).$$

That is,

$$Pf(\omega) = \left(\frac{I + T_3 + T_3^2}{3}\right) f(\omega) \text{ and } T_3^3 f(\omega) = f(\omega).$$

Similarly, if $\omega \in D_{12}$, D_{13} or D_{14} we have

$$Pf(\omega) = \left(\frac{I + T_3 + T_3^2}{3}\right) f(\omega) \text{ and } T_3^3 f(\omega) = f(\omega).$$

If $\omega \in D_{15}$ or D_{16}, we get

$$Pf(\omega) = \left(\frac{I + T_2 + T_3}{3}\right) f(\omega) = \left(\frac{I + T_2T_3 + (T_2T_3)^2}{3}\right) f(\omega)$$

and $(T_2T_3)^3 f(\omega) = f(\omega)$. The cases of $\omega \in D_2$ or D_3 is similar.

We now define

$$u(\omega) = \begin{cases} u_1(\omega) & \text{if } \ \omega \in A_1 \\ u_3(\omega) & \text{if } \ \omega \in D_{1i} \\ u_1(\omega)u_3(\phi_1(\omega)) & \text{if } \ \omega \in D_{2j} \\ u_1(\omega) & \text{if } \ \omega \in D_{3k} \end{cases}$$

and

$$\phi(\omega) = \begin{cases} \phi_1(\omega) & \text{if } \ \omega \in A_1 \\ \phi_3(\omega) & \text{if } \ \omega \in D_{1i} \\ \phi_3 \circ \phi_1(\omega) & \text{if } \ \omega \in D_{2j} \\ \phi_1(\omega) & \text{if } \ \omega \in D_{3k} \end{cases}$$

Define $Tf(\omega) = u(\omega)f(\phi(\omega))$. We have seen earlier that any limit point of D_{ij} belong to $A \cup D_{ij}$, it follows that u is continuous and ϕ is a homeomorphism. Hence, the proof of Theorem 3.1.2 (b) is complete. □

CHAPTER 4

Structure of Generalized 3-circular Projections on $\mathbb{C}^n$ and on Some Spaces of Matrices

In this chapter we will find the structures of generalized 3-circular projections on $\mathbb{C}^n$ with symmetric norm, and on some spaces of matrices with unitarily invariant norm and unitary congruence invariant norm.

Let G be a closed subgroup of $\mathcal{G}(X)$. We recall that a norm $\|\cdot\|$ on a Banach space X is said to be G-invariant if

$$\|g(x)\| \;=\; \|x\| \quad \forall\, g \in G,\; x \in X.$$

Most of the contents of this chapter are from [3].

4.1 Symmetric norms

We start with the following lemma.

Lemma 4.1.1. *Let $P_0 + \lambda_1 P_1 + \lambda_2 P_2 = T$, where $T \in \mathcal{G}(X)$ and λ_1, λ_2, P_1 and P_2 are as in Definition 1.2.1. If P_1 is a bi-circular projection, then so is P_0.*

Proof. We first note that if P_1 is a bi-circular projection, then so is $I - P_1 = P_0 + P_2$. As every bi-circular projection is hermitian, $e^{i\theta(P_0+P_2)}$ is an isometry for all $\theta \in \mathbb{R}$. Suppose there exists $0 \neq x \in X$ such that $\|e^{i\theta P_0}x\| < \|x\|$. Then we have

$$\|x\| = \|e^{i\theta(P_0+P_2)}x\| = \|e^{i\theta P_2}e^{i\theta P_0}x\| \leq \|e^{i\theta P_0}x\| < \|x\|,$$

a contradiction. $\square$

In this section we will find the structures of generalized 3-circular projections on $\mathbb{C}^n$ with a symmetric norm.

The isometry group of a given symmetric norm (see Theorem 1.2.23) is the group of generalized permutation matrices, that is, matrices of the form $T = DR$, where D is a diagonal matrix with entries from the unit circle and P is a permutation matrix. We will denote this group by G.

Remark 4.1.2. *Let R be a permutation matrix such that the permutation associated with R fixes m elements, $m \geq 0$, and has k disjoint cycles of lengths $n_1, n_2, \ldots, n_k$. Let π_j be the cycle $(1\ 2\ \ldots\ j-1\ j)$ and R_j the permutation matrix for the cycle π_{n_j}, $j = 1, 2, \ldots, k$. Then R is permutationally similar to $R_1 \oplus R_2 \oplus \cdots \oplus R_k \oplus I_m$.*

Theorem 4.1.3. *Let $\|\cdot\|$ be a symmetric norm on $\mathbb{C}^n$ and P_0 a generalized 3-circular projection. Then one and only one of the following assertions holds:*

(a) P_0 is a bi-circular projection.

(b) There exist $m \geq 0$, $k \geq 1$, projections $P_{0,i}$, $i = 0, \ldots, k$ such that P_0 is permutationally similar to $P_{0,1} \oplus P_{0,2} \oplus \cdots \oplus P_{0,k} \oplus P_{0,0}$, where

$$P_{0,i} = \frac{1}{3}\begin{pmatrix} 1 & d_{i1} & d_{i1}d_{i2} \\ d_{i2}d_{i3} & 1 & d_{i2} \\ d_{i3} & d_{i1}d_{i3} & 1 \end{pmatrix} \text{ and } P_{0,0} = diag(p_1, p_2, \ldots, p_m)$$

with $p_j \in \{0, 1\}$ for all $j = 1, 2, \ldots, m$ and $d_{i1}d_{i2}d_{i3} = 1$.

Proof. Suppose $P_0 + \lambda_1 P_1 + \lambda_2 P_2 = T$ where λ_i and P_i, $i = 1, 2$ are as in Definition 1.2.1. Since the isometry group for the symmetric norm is G, $T = DR$ where D is a diagonal matrix whose elements are of unit modulus and R is a permutation matrix. By Remark 4.1.2, R is permutationally similar to $R_1 \oplus R_2 \oplus \cdots \oplus R_k \oplus I_m$. We write $D = D_1 \oplus D_2 \oplus \cdots \oplus D_k \oplus D_0$ accordingly. Then T will be permutationally similar to

$$\begin{pmatrix} D_1R_1 & 0 & \cdots & 0 \\ 0 & D_2R_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & D_kR_k & 0 \\ 0 & 0 & \cdots & D_0 \end{pmatrix} \tag{4.1.1}$$

By Lemma 3.2.4, the eigenvalues of T are $\{1, \lambda_1, \lambda_2\}$ which is the union of eigenvalues of D_iR_i, $i = 1, 2, \ldots, k$ and D_0. We also note that D_iR_i has n_i distinct eigenvalues.

Suppose $k = 0$. Then $T = D$ and from Lemma 3.2.4,

$$P_0 = \frac{(D - \lambda_1 I)(D - \lambda_2 I)}{(1 - \lambda_1)(1 - \lambda_2)}.$$

Hence, P_0 is a diagonal matrix whose elements are 0 or 1. This implies that for any $\lambda \in \mathbb{T} \setminus \{1\}$, $P_0 + \lambda(I - P)$ is a diagonal matrix with entries 1 or λ and hence an isometry. Thus, P_0 is a bi-circular projection and assertion (a) follows.

Suppose $k > 0$. Then Equation 3.2.1 implies that

$$(D_iR_i - I_{n_i})(D_iR_i - \lambda_1 I_{n_i})(D_iR_i - \lambda_2 I_{n_i}) = 0$$

$\forall\ i = 1, 2, \ldots, k$ and

$$(D_0 - I_m)(D_0 - \lambda_1 I_m)(D_0 - \lambda_2 I_m) = 0.$$

Hence, the eigenvalues of D_iR_i and D_0 are $\{1, \lambda_1, \lambda_2\}$. We again note that the eigenvalues of D_iR_i, $i = 1, \ldots, k$ are distinct. Therefore, we have $n_1 = n_2 = \cdots = n_k = 3$. Suppose $D_i = \operatorname{diag}(d_{i1}, d_{i2}, d_{i3})$. Then we have

$$D_iR_i \;=\; \begin{pmatrix} d_{i1} & 0 & 0 \\ 0 & d_{i2} & 0 \\ 0 & 0 & d_{i3} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & d_{i1} & 0 \\ 0 & 0 & d_{i2} \\ d_{i3} & 0 & 0 \end{pmatrix}$$

Since $tr(D_iR_i) = 0$, we have $1+\lambda_1+\lambda_2 = 0$. Thus, λ_1 and λ_2 are the cube roots of identity.

As T is permutationally similar to the matrix displayed in Equation 4.1.1 and $P_0 = \frac{I+T+T^2}{3}$, we obtain P_0 is permutationally similar to $P_{0,1} \oplus P_{0,2} \oplus \cdots \oplus P_{0,k} \oplus P_{0,0}$; where

$$P_{0,i} = \frac{I_3 + D_iR_i + (D_iR_i)^2}{3},$$

$i = 1, 2, \ldots, k$ and

$$P_{0,0} = \frac{I_m + D_0 + D_0^2}{3}.$$

This implies that

$$P_{0,i} = \frac{1}{3}\begin{pmatrix} 1 & d_{i1} & d_{i1}d_{i2} \\ d_{i2}d_{i3} & 1 & d_{i2} \\ d_{i3} & d_{i1}d_{i3} & 1 \end{pmatrix} \text{ and } P_{0,0} = \text{diag}(p_1, p_2, \ldots, p_m).$$

Here, $p_j \in \{0, 1\}$ for all $j = 1, 2, \cdots, m$.

Moreover, $det(D_iR_i) = d_{i,1}d_{i,2}d_{i,3} = 1$.

Thus, the proof of assertion (b) is complete. □

4.2 Unitarily invariant norms

In this section we will characterize generalized 3-circular projections on $\mathbb{M}_{m,n}(\mathbb{C})$ with a unitarily invariant norm.

From Theorem 1.2.25 we know that if $m \neq n$, then any isometry T is of the form $T(A) = UAV$ where $U \in U(\mathbb{C}^m)$ and $V \in U(\mathbb{C}^n)$. If $m = n$, then an isometry T on $\mathbb{M}_n(\mathbb{C})$ has the form either $T(A) = UAV$ or $T(A) = UA^tV$ where U, V are unitaries in $\mathbb{M}_n(\mathbb{C})$ and A^t denotes the transpose of a matrix A.

Remark 4.2.1. Let P_0 be a generalized 3-circular projection. Then $\exists$ λ_1, λ_2, P_1 and P_2 as in Definition 1.2.1 such that $P_0 + \lambda_1P_1 + \lambda_2P_2 = T$. By Lemma 3.2.4, T has spectrum $\{1, \lambda_1, \lambda_2\}$. Let us assume that U has eigenvalues $u_1, \ldots, u_m$ and V has eigenvalues $v_1, \ldots, v_n$. If $T(A) = UAV$, then identifying $\mathbb{M}_{m,n}(\mathbb{C})$ as $\mathbb{R}^m \otimes \mathbb{R}^n$, we see that $T(x \otimes y^t) = Ux \otimes y^t V$. Thus, T has eigenvalues $u_i v_j$, $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$.

Theorem 4.2.2. *Let $\|\cdot\|$ be a unitarily invariant norm on $\mathbb{M}_{m,n}(\mathbb{C})$ and P_0 a generalized 3-circular projection such that the isometry associated with it is of the form $A \longmapsto UAV$ for some $U \in U(\mathbb{C}^m)$ and $V \in U(\mathbb{C}^n)$. Suppose $\lambda_1 + \lambda_2 = -1$, then there exist $R_i = R_i^* = R_i^2$ in $\mathbb{M}_m(\mathbb{C})$ and $S_i = S_i^* = S_i^2$ in $\mathbb{M}_n(\mathbb{C})$, $i = 0, 1, 2$ such that P_0 has the form $A \longmapsto R_0AS_0 + R_1AS_1 + R_2AS_2$.*

Proof. Let $P_0 + \lambda_1 P_1 + \lambda_2 P_2 = T$ where λ_1, λ_2, P_1 and P_2 are as in Definition 1.2.1. We first note that $1 + \lambda_1 + \lambda_2 = 0$ which implies that λ_1 and λ_2 are cube roots of unity. By Lemma 3.2.4, T has spectrum $\{1, \omega, \omega^2\}$. Assume that U has eigenvalues $u_1, \ldots, u_m$ and V has eigenvalues $v_1, \ldots, v_n$. Then by Remark 4.2.1, T has eigenvalues $u_i v_j$, $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$. Without loss of generality we may assume that $u_1 = v_1 = 1$. Thus, the spectrum of U and V is a subset of $\{1, \omega, \omega^2\}$. Hence, we have $U^3 = I$ and $V^3 = I$. Let

$$R_i = \frac{I + \mu_i U + \mu_i^2 U^2}{3} \quad \text{and} \quad S_i = \frac{I + \mu_i V^* + \mu_i^2 (V^*)^2}{3},$$

where $i = 0, 1, 2$; $\mu_0 = 1$, $\mu_1 = \omega$ and $\mu_2 = \omega^2$. So, we have

$$R_i^* = \frac{I + \overline{\mu_i} U^* + \overline{\mu_i^2} (U^2)^*}{3} = \frac{I + \mu_i^2 U^2 + \mu_i U}{3} = R_i,$$

because $U^3 = I$ and μ_i's are cube roots of unity.

Further, we have

$$\begin{aligned} R_i^2 &= \frac{I + \mu_i^2 U^2 + \mu_i^4 U^4 + 2\mu_i U + 2\mu_i^2 U^2 + 2\mu_i^3 U^3}{9} \\ &= \frac{3I + 3\mu_i U + 3\mu_i^2 U^2}{9} \qquad (\text{using } U^3 = I,\ \mu_i^3 = 1) \\ &= R_i. \end{aligned}$$

Similarly, we will get $S_i = S_i^* = S_i^2$.

Finally, we observe that

$$P_0(A) = R_0AS_0 + R_1AS_1 + R_2AS_2.$$ □

Theorem 4.2.3. *Let $\|\cdot\|$ be a unitarily invariant norm on $\mathbb{M}_{m,n}(\mathbb{C})$ and P_0 a generalized 3-circular projection such that the isometry associated with P_0 is of the form $A \longmapsto UAV$ for some $U \in U(\mathbb{C}^m)$ and $V \in U(\mathbb{C}^n)$. Suppose $\lambda_1 + \lambda_2 \neq -1$, then one and only one of following assertions holds:*

(*a*) *There exists $R \in \mathbb{M}_n(\mathbb{C})$ with $R = R^* = R^2$ such that $P_0(A) = AR$, or there exists $S \in \mathbb{M}_m(\mathbb{C})$ with $S = S^* = S^2$ such that $P_0(A) = SA$. In both cases, P_0 is a bi-circular projection.*

(*b*) $\lambda_i^2 = \lambda_j$, *$i, j = 1, 2$ and $i \neq j$;*

(*b*1) *λ_1 is of order p and λ_2 is of order q with $p = 2q$. In this case we have one of the following conditions:*

(*i*) *P_0 is a bi-circular projection.*

(*ii*) *P_1 is generalized bi-circular projection and $(\lambda_1)^{p/2} = (\lambda_2)^{q/2} = -1$. Moreover, P_0 has the form*

$$A \longmapsto \frac{\lambda_1 A}{2(\lambda_1 - 1)} + \frac{UAV}{1 - \lambda_1^2} + \frac{\lambda_1 U^q A V^q}{2(1 + \lambda_1)}.$$

(*b*2) *$\lambda_i = \sqrt{\lambda_j}$ and λ_1, λ_2 are of order p, where p is an odd integer greater or equal to 5. Moreover, there exist $R_i = R_i^* = R_i^2$ in $\mathbb{M}_m(\mathbb{C})$ and $S_i = S_i^* = S_i^2$ in $\mathbb{M}_n(\mathbb{C})$ such that*

$$P_0(A) = \sum_{i=0}^{p-1} R_i A S_i,$$

where $i = 0, 1, \ldots, p - 1$.

(*c*) *$\lambda_1 \lambda_2 = 1$ and P_0 will have the same form as in (b2).*

Proof. Let $P_0 + \lambda_1 P_1 + \lambda_2 P_2 = T$ where λ_1, λ_2, P_1 and P_2 are as in Definition 1.2.1. Lemma 3.2.4 implies that T has spectrum $\{1, \lambda_1, \lambda_2\}$. Proceeding in the same way as in the beginning of proof of Theorem 4.2.2 we can say that the spectra of U and V are any one of the following sets:

$$\{1\},\ \{1, \lambda_1\},\ \{1, \lambda_2\} \text{ or } \{1, \lambda_1, \lambda_2\}.$$

So, we will have the following three cases.

Case I

Suppose the spectrum of U is $\{1\}$.

If the spectrum of V is $\{1\}$, $\{1, \lambda_1\}$ or $\{1, \lambda_2\}$ then the spectrum of T is $\{1\}$, $\{1, \lambda_1\}$ or $\{1, \lambda_2\}$ respectively, a contradiction.

Therefore, the spectrum of V will be $\{1, \lambda_1, \lambda_2\}$. In this case, $U = I$ and $T(A) = AV$. From Lemma 3.2.4, we have

$$P_0 A = \frac{(T - \lambda_1)(T - \lambda_2 I)A}{(1 - \lambda_1)(1 - \lambda_2)}.$$

We define

$$R = \frac{(V - \lambda_1 I)(V - \lambda_2 I)}{(1 - \lambda_1)(1 - \lambda_2)}.$$

Then we obtain $P_0(A) = AR$. We claim that $R = R^*$. To see the claim, from the expression of R we have

$$V^2 = (1 - \lambda_1)(1 - \lambda_2)R + (\lambda_1 + \lambda_2)V - \lambda_1\lambda_2 I. \tag{4.2.1}$$

Equation (4.2.1) implies that

$$(V^*)^2 = (V^2)^* = (1 - \overline{\lambda_1})(1 - \overline{\lambda_2})R^* + (\overline{\lambda_1} + \overline{\lambda_2})V^* - \overline{\lambda_1\lambda_2} I. \tag{4.2.2}$$

From Equation (3.2.1), we have $(V - I)(V - \lambda_1 I)(V - \lambda_2 I) = 0$, that is,

$$V^3 - (1 + \lambda_1 + \lambda_2)V^2 + (\lambda_1 + \lambda_2 + \lambda_1\lambda_2)V - \lambda_1\lambda_2 I = 0.$$

Multiplying both sides by V^* we get

$$V^2 - (1 + \lambda_1 + \lambda_2)V + (\lambda_1 + \lambda_2 + \lambda_1\lambda_2)I - \lambda_1\lambda_2 V^* = 0. \tag{4.2.3}$$

Multiplying again by V^* we have

$$V - (1 + \lambda_1 + \lambda_2)I + (\lambda_1 + \lambda_2 + \lambda_1\lambda_2)V^* - \lambda_1\lambda_2 (V^*)^2 = 0. \tag{4.2.4}$$

Adding Equations (4.2.3) and (4.2.4), we get

$$V^2 - (\lambda_1 + \lambda_2)V + (\lambda_1\lambda_2 - 1)I + (\lambda_1 + \lambda_2)V^* = \lambda_1\lambda_2 (V^*)^2.$$

Substituting V^2 and $(V^*)^2$ from Equations (4.2.1) and (4.2.2), we get $R = R^*$. Moreover, as P_0 is a projection we get $R^2 = R$.

We observe that for any $\mu \in \mathbb{T} \setminus \{1\}$,

$$\begin{aligned}
[P_0 + \mu(I - P_0)]A &= P_0A + \mu(A - P_0A) \\
&= AR + \mu(A - AR) \\
&= A[R + \mu(I - R)] \\
&= AW,
\end{aligned}$$

where $W = R + \mu(I - R)$. To show that $P_0 + \mu(I - P_0)$ is an isometry, we need to show that W is a unitary. We consider

$$\begin{aligned}
WW^* &= [R + \mu(I - R)][R^* + \overline{\mu}(I - R^*)] \\
&= [R + \mu(I - R)][R + \overline{\mu}(I - R)] \\
&= R + I - R \\
&= I = W^*W.
\end{aligned}$$

Therefore, P_0 is a bi-circular projection.

Hence, assertion (a) is proved.

Case II

Suppose the spectrum of U is $\{1, \lambda_1\}$.

The case in which the spectrum of U is $\{1, \lambda_2\}$ is exactly similar.

So, the choices of spectrum of V are $\{1, \lambda_1\}$, $\{1, \lambda_2\}$ or $\{1, \lambda_1, \lambda_2\}$.

(A) If the spectrum of V is $\{1, \lambda_1\}$, then T will have spectrum $\{1, \lambda_1, \lambda_1^2\}$. This implies that $\lambda_1^2 = \lambda_2$.

Let p and q be the order of λ_1 and λ_2 respectively. Then we have $\lambda_1^{2q} = \lambda_2^q = 1$ and $\lambda_2^p = \lambda_1^{2p} = 1$. This implies that p divides $2q$ and q divides p or $2q = k_1p$ and $p = k_2q$ for some positive integers k_1 and k_2. Thus, we have $k_1k_2 = 2$. So, either $k_1 = 1$, $k_2 = 2$ or $k_1 = 2$, $k_2 = 1$.

If $k_1 = 1$ and $k_2 = 2$ we get $p = 2q$.

If $k_1 = 2$ and $k_2 = 1$ we get $p = q$.

Suppose $p = 2q$. Then we have

$$\begin{aligned} T^q &= P_0 + \lambda_1^q P_1 + \lambda_2^q P_2 \\ &= P_0 + \lambda_1^q P_1 + P_2. \end{aligned}$$

It follows that P_1 is a GBP. Proposition 1.2.26 implies that P_1 is either a bi-circular projection or $\lambda_1^q = -1$.

If P_1 is a bi-circular projection, then by Remark 4.1.1 we conclude that P_0 is also a bi-circular projection.

Hence, assertion (i) is proved.

If P_1 is not a bi-circular projection, then we have $\lambda_1^q = -1$.

Suppose $\lambda_1 = \sqrt{\lambda_2}$, then we get $(\lambda_2)^{q/2} = -1$.

Suppose $\lambda_1 = -\sqrt{\lambda_2}$, then we get $(-1)^q(\lambda_2)^{q/2} = -1$. This shows that $(\lambda_2)^{q/2} = -1$, otherwise if $(\lambda_2)^{q/2} = 1$ then we will get $\lambda_1^q = 1$, which is a contradiction.

So, in both cases we have $(\lambda_2)^{q/2} = -1$. Since $q = p/2$ we also have $\lambda_1^{p/2} = -1$.

For the form of P_0 we consider the following three equations,

$$\begin{aligned} P_0 - P_1 + P_2 &= T^q \\ P_0 + \lambda_1 P_1 + \lambda_2 P_2 &= T \\ P_0 + P_1 + P_2 &= I. \end{aligned}$$

Eliminating P_1 and P_2 we get

$$P_0 = \frac{\lambda_1 I}{2(\lambda_1 - 1)} + \frac{T}{1 - \lambda_1^2} + \frac{\lambda_1 T^q}{2(1 + \lambda_1)}. \tag{4.2.5}$$

Hence, assertion (ii) is proved.

Suppose $p = q$ and $\lambda_1 = \pm\sqrt{\lambda_2}$.

If $\lambda_1 = -\sqrt{\lambda_2}$, then we have $\lambda_1^p = (-\sqrt{\lambda_2})^p = 1$ or $(-1)^p(\lambda_2)^{p/2} = 1$. This shows that p is odd, otherwise $(\lambda_2)^{p/2} = 1$, a contradiction because the order of λ_2 is p. Hence, we get $(\lambda_2)^{p/2} = -1$. It follows that $\lambda_1^p = -1$, a contradiction since the order of λ_1 is p.

If $\lambda_1 = \sqrt{\lambda_2}$, then we have $\lambda_1^p = (\sqrt{\lambda_2})^p = (\lambda_2)^{p/2} = 1$. This again implies that that p is odd. Since, we have $\lambda_1 + \lambda_2 \neq -1$ we conclude that $p > 3$. As the order of λ_1 is p, we have $U^p = I$ and $V^p = I$. Let

$$R_i = \sum_{j=0}^{p-1} \frac{\mu_i^j U^j}{p} \quad \text{and} \quad S_i = \sum_{j=0}^{p-1} \frac{\overline{\mu_i}^j V^j}{p},$$

where $i = 0, 1, \ldots, p-1$ and $\mu_0 = 1, \mu_1, \ldots, \mu_{p-1}$ are the p distinct roots of unity. We also observe that $R_i = R_i^* = R_i^2$ and $S_i = S_i^* = S_i^2$. Thus, we conclude that P_0 is of the form

$$A \longmapsto \sum_{i=0}^{p-1} R_i A S_i.$$

Hence, assertion $(b2)$ is proved.

(B) If the spectrum of V is $\{1, \lambda_2\}$, then T will have spectrum $\{1, \lambda_1, \lambda_2, \lambda_1\lambda_2\}$. This implies that $\lambda_1\lambda_2 = 1$ and hence λ_1 and λ_2 are of the same order.

Now, we have

$$\begin{aligned} T &= P_0 + \lambda_1 P_1 + \overline{\lambda_1} P_2 \\ \implies \lambda_1 T &= P_2 + \lambda_1 P_0 + \lambda_1^2 P_1. \end{aligned}$$

Because $\lambda_1 T$ is again an isometry, we are reduced to the previous case and we get assertion (c).

(C) If the spectrum of V is $\{1, \lambda_1, \lambda_2\}$, then T will have spectrum $\{1, \lambda_1, \lambda_2, \lambda_1\lambda_2, \lambda_1^2\}$. This implies that $\lambda_1\lambda_2 = 1$ and $\lambda_1^2 = \lambda_2$. Therefore, we have $\lambda_1^3 = \lambda_2^3 = 1$, a contradiction since $\lambda_1 + \lambda_2 \neq -1$.

Case III

Suppose that the spectrum of U is $\{1, \lambda_1, \lambda_2\}$.

(A) If the spectrum of V is $\{1\}$, then $V = I$. We proceed in the same way as in **Case I** to get $S \in \mathbb{M}_m(\mathbb{C})$ such that $S = S^* = S^2$ and $P_0 A = SA$. Thus, P_0 is a bi-circular projection.

(B) If the spectrum of V is $\{1, \lambda_1\}$ or $\{1, \lambda_2\}$, then we proceed exactly as the case in which the spectrum of U is $\{1, \lambda_1\}$ and of V is $\{1, \lambda_1, \lambda_2\}$.

(C) If the spectrum of V is $\{1, \lambda_1, \lambda_2\}$, then the spectrum of T will be $\{1, \lambda_1, \lambda_2, \lambda_1\lambda_2, \lambda_1^2, \lambda_2^2\}$. Thus, we have $\lambda_1\lambda_2 = 1$, $\lambda_1^2 = \lambda_2$ and $\lambda_2^2 = \lambda_1$. Hence, $1 = \lambda_1\lambda_2 = \lambda_1\lambda_1^2 = \lambda_1^3$. Similarly, we have $\lambda_2^3 = 1$. Thus, we get λ_1 and λ_2 are cube roots of unity, a contradiction. □

We now give an example to demonstrate assertion $(b1)(ii)$ of the above theorem.

Example 4.2.4. *We start with a GBP P_1 on $\mathbb{M}_3(\mathbb{C})$ with a unitarily invariant norm such that $P_1 = \frac{I+S}{2}$. Here, S is an isometry on $\mathbb{M}_3(\mathbb{C})$ given by $S(A) = UAV$, where*

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

For $A = (a_{ij}) \in \mathbb{M}_3(\mathbb{C})$, we define projections P_0 and P_2 as follows:

$$P_0(A) = \begin{pmatrix} 0 & 0 & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_2(A) = \begin{pmatrix} 0 & 0 & a_{13} \\ 0 & 0 & 0 \\ 0 & 0 & a_{33} \end{pmatrix}.$$

Then we have $P_0 \oplus P_1 \oplus P_2 = I$.

Let T be an isometry on $\mathbb{M}_3(\mathbb{C})$ defined as $T(A) = WAZ$, where

$$W = \begin{pmatrix} i\omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i\omega \end{pmatrix} \text{ and } Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i\omega \end{pmatrix}.$$

Then we have $T = P_0 + \lambda_1 P_1 + \lambda_2 P_2$, where $\lambda_1 = i\omega$ and $\lambda_2 = -\omega^2$.

Theorem 4.2.5. *Let $\|\cdot\|$ be a unitarily invariant norm on $\mathbb{M}_n(\mathbb{C})$ and P_0 a generalized 3-circular projection such that the isometry associated with P_0 is of the form $A \longmapsto UA^tV$ for some $U,\ V \in U(\mathbb{C}^n)$. Then one and only one of the following assertions holds:*

(*a*) $\lambda_1^2+\lambda_2^2 = -1$ *and there exist* $R_i = R_i^* = R_i^2$ *and* $S_i = S_i^* = S_i^2$ *in* $\mathbb{M}_n(\mathbb{C})$, $i = 0, 1, 2$ *such that* P_0 *has the form* $A \longmapsto R_0AS_0 + R_1AS_1 + R_2AS_2$.

(*b*) $\lambda_i^4 = \lambda_j^2$, $i, j = 1, 2$ *and* $i \neq j$;

(b1) λ_1^2 *is of order* p *and* λ_2^2 *is of order* q *with* $p = 2q$. *In this case, we have one of the following conditions:*

(*i*) P_0 *is a bi-circular projection.*

(*ii*) P_1 *is generalized bi-circular projection and* $\lambda_1^p = \lambda_2^q = -1$. *Moreover,* P_0 *has the form*

$$A \longmapsto \frac{\lambda_1^2 A}{2(\lambda_1^2-1)} + \frac{UV^tAU^tV}{1-\lambda_1^4} + \frac{\lambda_1^2(UV^t)^qA(U^tV)^q}{2(1+\lambda_1^2)}.$$

(b2) $\lambda_i^2 = \lambda_j$; λ_1^2 *and* λ_2^2 *are of order* p, *where* p *is an odd integer greater or equal to* 5. *Moreover, there exist* $R_i = R_i^* = R_i^2$ *and* $S_i = S_i^* = S_i^2$ *in* $\mathbb{M}_n(\mathbb{C})$ *such that*

$$P_0(A) = \sum_{i=0}^{p-1} R_iAS_i,$$

where $i = 0, 1, \ldots, p-1$.

Proof. Let $P_0 + \lambda_1 P_1 + \lambda_2 P_2 = T$ where λ_1, λ_2, P_1 and P_2 are as in Definition 1.2.1. Now, as $T(A) = UA^tV$ we have

$$T^2(A) = P_0(A) + \lambda_1^2 P_1(A) + \lambda_2^2 P_2(A) = UV^tAU^tV.$$

By Lemma 3.2.4, the spectrum of T^2 is $\{1, \lambda_1^2, \lambda_2^2\}$. Let $X = UV^t$ and $Y = U^tV$. We observe that X and Y are unitary matrices. Since eigenvalues of X and $X^t = VU^t$ are same, and $y = U^tV$, X, Y have same eigenvalues. Let the eigenvalues of X be $\nu_1, \nu_2, \ldots, \nu_n$. Then the eigenvalues of T^2 is $\nu_i\nu_j$, $1 \leq i, j \leq n$. Without loss of generality we can assume that $\nu_3 = 1$. Hence, the spectrum of X is a subset of $\{1, \lambda_1^2, \lambda_2^2\}$. We assume $\nu_1 = \lambda_1^2$ and $\nu_2 = \lambda_2^2$.

Suppose $\nu_1 + \nu_2 = -1$. Then the the spectrum of X is a subset of $\{1, \omega, \omega^2\}$ and hence we have $X^3 = I$ and $Y^3 = I$. Let

$$R_i = \frac{I + \mu_i X + \mu_i^2 X^2}{3} \quad \text{and} \quad S_i = \frac{I + \mu_i Y^* + \mu_i^2 (Y^*)^2}{3},$$

where $i = 0, 1, 2$; $\mu_0 = 1$, $\mu_1 = \omega$ and $\mu_2 = \omega^2$. It can be easily verified that $R_i = R_i^* = R_i^2$ and $S_i = S_i^* = S_i^2$. Also, we have

$$P_0(A) = R_0 A S_0 + R_1 A S_1 + R_2 A S_2$$

and assertion (a) follows.

Now, we suppose that $\nu_1 + \nu_2 \neq -1$. Then the spectrum of X will be one of the following:

$$\{1\},\ \{1, \nu_1\},\ \{1, \nu_2\} \text{ or } \{1, \nu_1, \nu_2\}.$$

If the spectrum of X is $\{1\}$, then X is the identity matrix and so is Y. It follows that T^2 is the identity operator and $\lambda_1^2 = 1 = \lambda_2^2$. This implies that $\lambda_1 = \lambda_2 = -1$, a contradiction.

So, we consider the following two cases.

Case I

Suppose the spectrum of X is $\{1, \nu_1\}$.

The case in which the spectrum of X is $\{1, \nu_2\}$ is exactly similar.

So, the spectrum of T^2 is $\{1, \nu_1, \nu_1^2\}$. This implies that $\nu_1^2 = \nu_2$. We proceed in the same way as in part **(A)**, **Case II** of Theorem 4.2.3. We see that if ν_1 and ν_2 are of different order, then either P_0 is a bi-circular projection or P_1 is a GBP. If P_1 is a GBP, and the order of ν_1 and ν_2 is p and q respectively, then we have $p = 2q$. Moreover, P_0 will be of the form

$$A \longmapsto \frac{\nu_1 A}{2(\nu_1 - 1)} + \frac{XAY}{1 - \nu_1^2} + \frac{\nu_1 X^q A Y^q}{2(1 + \nu_1)}.$$

Hence, assertion $(b1)$ follows.

If ν_1 and ν_2 are of same order, say p, then by arguments similar to part **(A)**, **Case II** of Theorem 4.2.3, we can show that $\nu_1 = \sqrt{\nu_2}$ and p is an odd integer greater than or equal to 5, since $\nu_1 + \nu_2 \neq -1$. In this case we have $X^p = I$ and $Y^p = I$. Let

$$R_i = \sum_{j=0}^{p-1} \frac{\mu_i^j X^j}{p} \quad \text{and} \quad S_i = \sum_{j=0}^{p-1} \frac{\overline{\mu_i}^j Y^j}{p},$$

where $i = 0, 1, \ldots, p-1$ and $\mu_0 = 1, \mu_1, \ldots, \mu_{p-1}$ are the p distinct roots of unity. Clearly, we have that $R_i = R_i^* = R_i^2$ and $S_i = S_i^* = S_i^2$ and we conclude that P_0 is of the form

$$A \longmapsto \sum_{i=0}^{p-1} R_i A S_i.$$

Thus, assertion $(b2)$ is proved.

Case II

Suppose the spectrum of X is $\{1, \nu_1, \nu_2\}$, $1 \neq \nu_1 \neq \nu_2 \neq 1$, then the spectrum of T^2 is

$$\{1, \nu_1, \nu_2, \nu_1\nu_2, \nu_1^2, \nu_2^2\} = \{1, \nu_1, \nu_2\}.$$

We observe that $\nu_1\nu_2 \neq \nu_1$ or ν_2; otherwise $\nu_1 = 1$ or $\nu_2 = 1$. It follows that $\nu_1\nu_2 = 1$. We also see that $\nu_1^2 \neq 1$ or ν_1. If so, we will have $\nu_1 = \nu_2$ or $\nu_1 = 1$ respectively, both leading to a contradiction. So, the only possibility is that $\nu_1^2 = \nu_2$. Similarly, we can show that $\nu_2^2 = \nu_1$. But this implies that ν_1 and ν_2 are cube roots of unity, which contradicts our assumption that $\nu_1 + \nu_2 \neq -1$. □

4.3 Unitary congruence invariant norms

In this section we characterize generalized 3-circular projections on $S_n(\mathbb{C})$ with a unitary congruence invariant norm.

We recall Theorem 1.2.28, that is, for a unitary congruence invariant norm on $S_n(\mathbb{C})$, any isometry T is given by $T(A) = U^t A U$, where U is a unitary in $\mathbb{M}_n(\mathbb{C})$.

Remark 4.3.1. *Suppose $T : S_n(\mathbb{C}) \longrightarrow S_n(\mathbb{C})$ is defined by $T(A) = U^t A U$, where $U \in U(\mathbb{C}^n)$. Assume that U^t has eigenvalues $u_1, u_2, \ldots, u_n$ with eigenvectors $x_1, x_2, \ldots, x_n$. Then T has eigenvalues $u_i u_j$ with eigenvectors $x_i x_j^t + x_j x_i^t$ for $1 \leq i, j \leq n$. To see this, we first note that $x_i x_j^t + x_j x_i^t$ is a symmetric matrix. Then we see that*

$$T(x_i x_j^t + x_j x_i^t) \;=\; U^t (x_i x_j^t + x_j x_i^t) U$$

$$\begin{aligned} &= U^t x_i x_j^t U + U^t x_j x_i^t U \\ &= u_i x_i u_j x_j^t + u_j x_j u_i x_i^t \\ &= u_i u_j (x_i x_j^t + x_j x_i^t). \end{aligned}$$

So, if $i = j$ *we have* $T(x_i x_i^t) = U^t x_i x_i^t U = u_i^2 (x_i x_i^t)$.

Theorem 4.3.2. *Let* $\|\cdot\|$ *be a unitary congruence invariant norm on* $S_n(\mathbb{C})$ *and* P_0 *a generalized* 3*-circular projection. Then there exists* $U \in U(\mathbb{C}^n)$ *such that one and only one of the following assertions holds:*

(a) *U has three distinct eigenvalues. In this case,* $\lambda_1 + \lambda_2 = -1$*. Moreover, there exist* $R_i = R_i^* = R_i^2$ *in* $\mathbb{M}_n(\mathbb{C})$ *such that* P_0 *has the form* $A \longmapsto R_0^t A R_0 + R_1^t A R_2 + R_2^t A R_1$.

(b) *U has two distinct eigenvalues. In this case, one and only one of the following occurs:*

($b1$) $\lambda_i = \sqrt{\lambda_j}$, $i, j = 1, 2$ *and* $i \neq j$ *and* λ_i*'s are of order p, where p is an odd integer greater or equal to* 3*. Moreover, there exist* $R_i = R_i^* = R_i^2$ *and* $S_i = S_i^* = S_i^2$ *in* $\mathbb{M}_n(\mathbb{C})$ *such that*

$$P_0(A) = \sum_{i=0}^{p-1} R_i A S_i,$$

where $i = 0, 1, \ldots, p-1$.

($b2$) $\lambda_1 \lambda_2 = 1$ *and* P_0 *will have the same form as in* ($b1$).

Proof. Suppose T is of the form $A \longmapsto U^t A U$ for some $U \in U(\mathbb{C}^n)$. Let $P_0 + \lambda_1 P_1 + \lambda_2 P_2 = T$. By Lemma 3.2.4, T has spectrum $\{1, \lambda_1, \lambda_2\}$. Suppose U has eigenvalues $u_1, u_2, \ldots, u_n$. Then T has eigenvalues $u_i u_j$, $1 \leq i, j \leq n$.

We claim that U can have two or three distinct eigenvalues.

To see the claim, suppose that U has four distinct eigenvalues, say, u_1, u_2, u_3 and u_4. This implies that $u_1 u_2$, $u_1 u_3$, $u_1 u_4$ and u_1^2 are distinct eigenvalues of T which is impossible. Similarly, U cannot have more than four distinct eigenvalues.

If U has one eigenvalue, say, u then T will have eigenvalue u^2, which is a contradiction.

So, we consider the following two steps.

Step I

Suppose U has three distinct eigenvalues, say, u_1, u_2 and u_3.

This implies that

$$\{u_1^2, u_2^2, u_3^2, u_1u_2, u_2u_3, u_1u_3\} = \{1, \lambda_1, \lambda_2\}.$$

None of u_1u_2, u_2u_3 or u_1u_3 are equal and u_1^2 is equal to one and only one of u_2^2, u_3^2, u_2u_3 because if $u_1^2 = u_2^2 = u_3^2$, the spectrum of T will have have four distinct eigenvalues, namely, u_1u_2, u_2u_3, u_1u_3, u_1^2 which is not possible.

Suppose that $u_1^2 = u_2^2$. Then $u_3^2 = u_1u_2$. This implies that u_1^2, u_1u_2, u_2u_3, u_1u_3 are four distinct eigenvalues of T, which is impossible.

Therefore, we conclude that $u_1^2 = u_2u_3$, $u_2^2 = u_1u_3$ and $u_3^2 = u_1u_2$. Thus, we have $\{u_1^2, u_2^2, u_1u_2\} = \{1, \lambda_1, \lambda_2\}$.

Let $u_1^2 = \lambda_1$, $u_2^2 = \lambda_2$, $u_1u_2 = 1$ $(= u_3^2)$. Then for $i, j = 1, 2$; $i \neq j$

$$\lambda_i^3 = (u_i^2)^3 = (u_ju_3)^3 = u_j^2u_ju_3^3 = u_iu_3u_ju_3^3 = u_iu_ju_3^4 = u_3^2u_3^4 = 1.$$

Let $u_1^2 = 1$, $u_2^2 = \lambda_2$, $u_1u_2 = \lambda_1$. Then for the the triples $(i, j, k) = (1, 2, 3)$ or $(2, 3, 2)$ we have

$$\lambda_i^3 = (u_ju_1)^3 = u_j^2u_ju_1^3 = u_1u_ku_ju_1^3 = u_1u_1^2u_1^4 = 1.$$

So, we have λ_1 and λ_2 become the cube roots of unity and hence $T^3(A) = A = X^tAX$ for all $A \in S_n(\mathbb{C})$, where $X = U^3$.

Suppose $X = (x_{ij})$. By putting $A = E_{11}, E_{22}, \ldots, E_{nn}$ we get $x_{ii}^2 = 1$ for $i = 1, 2, \ldots, n$ and rest of the elements of X to be zero. Similarly, by putting $A = E_{12} + E_{21}, E_{13} + E_{31}, \ldots, E_{1n} + E_{n1}$ we get $x_{11} = x_{22} = \cdots = x_{nn}$. Therefore, we conclude that $X = I$ or $-I$.

Let $U^3 = I$. We put

$$R_i = \frac{I + \mu_iU + \mu_i^2U^2}{3},$$

where $i = 0, 1, 2$; $\mu_0 = 1, \mu_1 = \omega$ and $\mu_2 = \omega^2$. Then we have

$$P_0A = R_0^tAR_0 + R_1^tAR_2 + R_2^tAR_1.$$

Let $U^3 = -I$. We put

$$R_i = \frac{I - \mu_i U + \mu_i^2 U^2}{3},$$

where $i = 0, 1, 2$, $\mu_0 = 1, \mu_1 = \omega$ and $\mu_2 = \omega^2$. Then we obtain

$$P_0 A = R_0^t A R_0 + R_1^t A R_2 + R_2^t A R_1.$$

In both cases, we have $R_i = R_i^* = R_i^2$.

Step II

Suppose U has two distinct eigenvalues, say, u_1 and u_2, then the spectrum of T will be $\{u_1^2, u_2^2, u_1 u_2\} = \{1, \lambda_1, \lambda_2\}$.

Lemma 3.2.2 and Proposition 1.2.29 implies that λ_1 and λ_2 have the same order.

If $u_1^2 = 1$, $u_2^2 = \lambda_2$ and $u_1 u_2 = \lambda_1$, then we get $\lambda_1^2 = \lambda_2$. We proceed as in part **(A)**, **Case II** of Theorem 4.2.3 to get assertion $(b1)$. Here, we note that the order of λ_1 and λ_2 can be 3.

If $u_1^2 = \lambda_1$, $u_2^2 = \lambda_2$ and $u_1 u_2 = 1$, then we get $\lambda_1 \lambda_2 = 1$. We proceed as in part **(B)**, **Case II** of Theorem 4.2.3 to get assertion $(b2)$. $\square$

Chapter 5

Algebraic Reflexivity of the Set of Isometries of Order n

We recall the definition of algebraic reflexivity. A subset $\mathcal{S}$ of $B(X)$ is said to be algebraic reflexive if $\mathcal{S} = \overline{\mathcal{S}}^a$. The algebraic closure $\overline{\mathcal{S}}^a$ of $\mathcal{S}$ is defined as follows:

$T \in \overline{\mathcal{S}}^a$ if for every $x \in X$ there exists $T_x \in S$ such that $T(x) = T_x(x)$.

We also recall that $T \in \mathcal{G}^n(X) \Longleftrightarrow T \in \mathcal{G}(X)$ and $T^n = I$

In this chapter we prove that if $\mathcal{G}(C_0(\Omega, X))$ is algebraically reflexive, then $\mathcal{G}^n(C_0(\Omega, X))$ is algebraically reflexive. Here, Ω is a locally compact Hausdorff space and X is a Banach space with trivial centralizer. As a corollary to this, we show that the set of generalized bi-circular projections on $C(\Omega, X)$ is algebraically reflexive. This answers a question raised in [14].

5.1 Statement of results

We recall Theorem 1.2.13.

$T \in \mathcal{G}(C_0(\Omega, X))$ if and only if $\exists$ a homeomorphism $\phi : \Omega \longrightarrow \Omega$ and a map $u : \Omega \longrightarrow$

$\mathcal{G}(X)$, continuous in strong operator topology, such that

$$Tf(\omega) = u_\omega(f(\phi(\omega))), \quad \forall\ f \in C_0(\Omega),\ \omega \in \Omega.$$

Our first result is the following.

Theorem 5.1.1. *Let Ω be a locally compact Hausdorff space and X a Banach space which has the strong Banach-Stone property. If $\mathcal{G}(C_0(\Omega, X))$ is algebraically reflexive, then $\mathcal{G}^n(C_0(\Omega, X))$ is algebraically reflexive.*

Combining the above Theorem with Theorem 1.2.31 we immediately have the following corollary.

Corollary 5.1.2. *Let Ω be a first countable compact Hausdorff space and X a uniformly convex Banach space such that $\mathcal{G}(X)$ is algebraically reflexive. Then $\mathcal{G}^n(C(\Omega, X))$ is algebraically reflexive.*

We also have

Corollary 5.1.3. *Let Ω and X be as in Corollary 5.1.2. Furthermore, assume that X does not have any generalized bi-circular projections. Then the set of generalized bi-circular projections on $C(\Omega, X)$ is algebraically reflexive.*

5.2 Proof of results

We start with the following lemma.

Lemma 5.2.1. *$T \in \mathcal{G}^n(C_0(\Omega, X))$ if and only if $\exists$ a homeomorphism ϕ of Ω and a map $u : \Omega \longrightarrow \mathcal{G}(X)$ satisfying*

$$u_\omega \circ u_{\phi(\omega)} \circ \cdots \circ u_{\phi^{n-1}(\omega)} = I, \quad \phi^n(\omega) = \omega, \quad \forall\ \omega \in \Omega;$$

where I denotes the identity map on X and T is given by

$$Tf(\omega) = u_\omega(f(\phi(\omega))), \quad \forall\ f \in C_0(\Omega),\ \omega \in \Omega.$$

Proof. We first note that since $T \in \mathcal{G}(C_0(\Omega, X))$, $\exists$ a homeomorphism $\phi : \Omega \longrightarrow \Omega$ and a map $u : \Omega \longrightarrow \mathcal{G}(X)$ such that

$$Tf(\omega) = u_\omega(f(\phi(\omega))), \ \ \forall\ f \in C_0(\Omega),\ \omega \in \Omega.$$

Secondly, as $T \in \mathcal{G}^n(C_0(\Omega, X))$ we have $T^n f(\omega) = f(\omega)$. This show that

$$u_\omega \circ u_{\phi(\omega)} \circ \cdots \circ u_{\phi^{n-1}(\omega)}(f(\phi^n(\omega))) = f(\omega). \tag{5.2.1}$$

For any fixed $x \in X$ and fixed $\omega \in \Omega$ we consider a function $f_x \in C_0(\Omega, X)$ such that $f_x(\omega) = x$. Applying Equation 5.2.1 to f_x we get

$$u_\omega \circ u_{\phi(\omega)} \circ \cdots \circ u_{\phi^{n-1}(\omega)}(x) = x.$$

Since this can be done for each $x \in X$ and each $\omega \in \Omega$ we conclude

$$u_\omega \circ u_{\phi(\omega)} \circ \cdots \circ u_{\phi^{n-1}(\omega)} = I.$$

This also implies that $f(\phi^n(\omega)) = f(\omega)$ for all $f \in C_0(\Omega, X)$. Hence, we get $\phi^n(\omega) = \omega$. ☐

Proof of Theorem 5.1.1

Let $T \in \overline{\mathcal{G}^n(C_0(\Omega, X))}^a$. Then for each $f \in C_0(\Omega, X)$ we have $Tf(\omega) = u^f_\omega(f(\phi_f(\omega)))$ where $u^f : \Omega \longrightarrow \mathcal{G}(X)$ is continuous in strong operator topology and satisfies

$$u^f_\omega \circ u^f_{\phi_f(\omega)} \circ \cdots \circ u^f_{\phi_f^{n-1}(\omega)} = I,$$

and ϕ_f is a homeomorphism of Ω such that $\phi^n_f(\omega) = \omega$ for all $\omega \in \Omega$. In particular $T \in \overline{\mathcal{G}(C_0(\Omega, X))}^a$. By the algebraically reflexivity of $\mathcal{G}(C_0(\Omega, X))$, we conclude that T is a surjective isometry on $C_0(\Omega, X)$ and hence $\exists$ a homeomorphism $\phi : \Omega \longrightarrow \Omega$ and a map $u : \Omega \longrightarrow \mathcal{G}(X)$ such that

$$Tf(\omega) = u_\omega(f(\phi(\omega))), \ \ \forall\ f \in C_0(\Omega),\ \omega \in \Omega.$$

To show that $\mathcal{G}^n(C_0(\Omega, X))$, we need to prove that $T^n = I$, that is, by Lemma 5.2.1

$$u_\omega \circ u_{\phi(\omega)} \circ \cdots \circ u_{\phi^{n-1}(\omega)} = I \text{ and } \phi^n(\omega) = \omega, \ \ \forall\ \omega \in \Omega.$$

Suppose $f = h \otimes x$, where h is a strictly positive function in $C_0(\Omega)$ and $0 \neq x \in X$. Then we have

$$\begin{aligned}
Tf(\omega) &= u_\omega^f(f(\phi_f(\omega))) = u_\omega(f(\phi(\omega))) \\
&\implies u_\omega^f(h(\phi_f(\omega))x) = u_\omega(h(\phi(\omega))x) \\
&\implies \|u_\omega^f(h(\phi_f(\omega))x)\| = \|u_\omega(h(\phi(\omega))x)\| \\
&\implies \|h(\phi_f(\omega))x\| = \|h(\phi(\omega))x\| \quad (\because\ u_\omega^f \text{ and } u_\omega \text{ are isometries}) \\
&\implies h(\phi_f(\omega)) = h(\phi(\omega)) \quad (\because\ h \text{ is strictly positive}) \\
&\implies u_\omega^f(x) = u_\omega(x).
\end{aligned}$$

Hence, we have $u_\omega^f = u_\omega$ for all $\omega \in \Omega$.

Let ω be any point in Ω. We consider the following cases.

Case I

Assume that $\omega = \phi(\omega)$. Then we have

$$\phi^n(\omega) = \phi(\phi(\cdots(\phi(\omega))\cdots)) \ (n \text{ times}) = \omega.$$

We choose $h \in C_0(\Omega)$ such that $0 < h(\omega) \leq 1$ and $h^{-1}(1) = \{\omega\}$. For $f = h \otimes x$, $0 \neq x \in X$, evaluating Tf at ω we get

$$\begin{aligned}
Tf(\omega) &= u_\omega(f(\phi(\omega))) = u_\omega^f(f(\phi_f(\omega))) \\
&\implies u_\omega(h(\phi(\omega))x) = u_\omega^f(h(\phi_f(\omega))x) \\
&\implies u_\omega(x) = u_\omega^f(h(\phi_f(\omega))x) \quad (\because\ h(\phi(\omega)) = h(\omega) = 1) \\
&\implies \|u_\omega(x)\| = \|u_\omega^f(h(\phi_f(\omega))x)\| \\
&\implies h(\phi_f(\omega)) = 1 \quad (u_\omega \text{ and } u_\omega^f \text{ are isometries}) \\
&\implies \phi_f(\omega) = \omega \quad (\text{by the choice of } h) \\
&\implies \phi_f^2(\omega) = \cdots = \phi_f^{n-1}(\omega) = \omega.
\end{aligned}$$

So, we have

$$I = u_\omega^f \circ u_{\phi_f(\omega)}^f \circ \cdots \circ u_{\phi_f^{n-1}(\omega)}^f$$

$$\begin{aligned} &= u_\omega^f \circ u_\omega^f \circ \cdots \circ u_\omega^f \\ &= u_\omega \circ u_\omega \circ \cdots \circ u_\omega \quad (\text{as } u_\omega^f = u_\omega). \end{aligned}$$

Case II

We assume that $\phi(\omega) \neq \omega$, $\phi^m(\omega) = \omega$ such that m divides n and $\phi^s(\omega) \neq \omega$ for all $s < m$.

As m divides n, there exist some positive integer q such that $n = mq$. Therefore, we have

$$\phi^n(\omega) = \phi^{mq}(\omega) = \phi^m(\phi^m(\cdots(\phi^m(\omega)))\cdots) \ (q \text{ times}) = \omega.$$

We now choose $h \in C_0(\Omega)$ such that $1 \leq h(\omega) \leq m$ and

$$h^{-1}(1) = \{\omega\},\ h^{-1}(2) = \{\phi(\omega)\}, \ldots,\ h^{-1}(m) = \{\phi^{m-1}(\omega)\}.$$

Let $f = h \otimes x$ for $0 \neq x \in X$. Evaluating Tf at ω we get

$$\begin{aligned} Tf(\omega) &= u_\omega(f(\phi(\omega))) = u_\omega^f(f(\phi_f(\omega))) \\ &\implies u_\omega(h(\phi(\omega))x) = u_\omega^f(h(\phi_f(\omega))x) \\ &\implies u_\omega(2x) = u_\omega^f(h(\phi_f(\omega))x) \quad (\because\ h(\phi(\omega)) = 2) \\ &\implies \|u_\omega(2x)\| = \|u_\omega^f(h(\phi_f(\omega))x)\| \\ &\implies h(\phi_f(\omega)) = 2 \quad (u_\omega \text{ and } u_\omega^f \text{ are isometries}) \\ &\implies \phi_f(\omega) = \phi(\omega) \quad (\text{by the choice of } h). \end{aligned}$$

Similarly, by applying Tf at $\phi(\omega), \ldots,\ \phi^{m-1}(\omega)$ we get

$$\phi_f^p(\omega) = \phi^p(\omega), \qquad \text{for } 2 \leq p \leq m.$$

We note that $\phi_f^m(\omega) = \phi^m(\omega) = \omega$. It follows that

$$\phi_f^{m+1}(\omega) = \phi_f(\phi_f^m(\omega)) = \phi_f(\omega) = \phi(\omega) = \phi(\phi^m(\omega)) = \phi^{m+1}(\omega).$$

Thus, we have

$$\phi_f^p(\omega) = \phi^p(\omega), \qquad \text{for } m+1 \leq p \leq n-1.$$

Using the above and the fact that $u_\omega = u_\omega^f$ for all $\omega \in \Omega$, we have

$$\begin{aligned} u_\omega \circ u_{\phi(\omega)} \circ \cdots \circ u_{\phi^{n-1}(\omega)} &= u_\omega^f \circ u_{\phi_f(\omega)}^f \circ \cdots \circ u_{\phi_f^{n-1}(\omega)}^f \\ &= I. \end{aligned}$$

Case III

We assume that $\phi(\omega) \neq \omega$, $\phi^m(\omega) = \omega$ such that m does not divides n and $\phi^s(\omega) \neq \omega$ for all $s < m$.

Therefore, $\exists$ integers r and q such that $n = mq + r$, $0 < r < m$. We choose $h \in C_0(\Omega)$ such that $1 \leq h(\omega) \leq m$ and

$$h^{-1}(1) = \{\omega\},\ h^{-1}(2) = \{\phi(\omega)\}, \ldots,\ h^{-1}(m) = \{\phi^{m-1}(\omega)\}.$$

By applying Tf at $\omega, \phi(\omega), \ldots,\ \phi^{m-1}(\omega)$ and proceeding in the same way as in **Case II** we will get

$$\phi_f^p(\omega) = \phi^p(\omega), \qquad \text{for } 1 \leq p \leq n-1.$$

We now see that

$$\begin{aligned}
Tf(\phi^{n-1}(\omega)) &= u_{\phi^{n-1}(\omega)}(f(\phi^n(\omega))) = u^f_{\phi^{n-1}(\omega)}(f(\phi_f(\phi^{n-1}(\omega)))) \\
&\implies u_{\phi^{n-1}(\omega)}(h(\phi^n(\omega))x) = u^f_{\phi^{n-1}(\omega)}(h(\phi_f(\phi_f^{n-1}(\omega)))x) \\
&\implies u_{\phi^{n-1}(\omega)}(h(\phi^n(\omega))x) = u^f_{\phi^{n-1}(\omega)}(h(\phi_f^n(\omega))x) \\
&\implies u_{\phi^{n-1}(\omega)}(h(\phi^n(\omega))x) = u^f_{\phi^{n-1}(\omega)}(h(\omega)x) \quad (\because\ \phi_f^n(\omega) = \omega) \\
&\implies u_{\phi^{n-1}(\omega)}(h(\phi^n(\omega))x) = u^f_{\phi^{n-1}(\omega)}(x) \quad (\because\ h(\omega) = 1) \\
&\implies \|u_{\phi^{n-1}(\omega)}(h(\phi^n(\omega))x)\| = \|u^f_{\phi^{n-1}(\omega)}(x)\| \\
&\implies h(\phi^n(\omega)) = 1 \quad (u_{\phi^{n-1}(\omega)} \text{ and } u^f_{\phi^{n-1}(\omega)} \text{ are isometries}) \\
&\implies \phi^n(\omega) = \omega \quad (\text{by the choice of } h).
\end{aligned}$$

But, our assumption that $\phi^m(\omega) = \omega$ implies that $\phi^{mq}(\omega) = \omega$. Hence, we have

$$\omega = \phi^n(\omega) = \phi^{r+mq}(\omega) = \phi^r(\phi^{mq}(\omega)) = \phi^r(\omega),$$

a contradiction because $r < m$.

Case IV

We assume that $\omega,\ \phi(\omega), \ldots,\ \phi^{n-1}(\omega)$ are all distinct.

Choose $h \in C_0(\Omega)$ such that $1 \leq h(\omega) \leq n$ and

$$h^{-1}(1) = \{\omega\},\ h^{-1}(2) = \{\phi(\omega)\}, \ldots,\ h^{-1}(n) = \{\phi^{n-1}(\omega)\}.$$

Proceeding the same way as in **Case III** we get

$$\phi^n(\omega) = \omega \text{ and } u_\omega \circ u_{\phi(\omega)} \circ \cdots \circ u_{\phi^{n-1}(\omega)} = I.$$

This completes the proof of Theorem 5.1.1. □

Proof of Corollary 5.1.3

We denote the set of all generalized bi-circular projections on $C(\Omega, X)$ by $\mathcal{P}$. Let $P \in \overline{\mathcal{P}}^a$. Then for each $f \in C(\Omega, X)$, there exist $P_f \in \mathcal{P}$ such that $Pf = P_f f$. Therefore, by Theorem 2.3.2 and the assumption on X, for each f there exists a homeomorphism ϕ_f of Ω, $u^f : \Omega \longrightarrow \mathcal{G}(X)$ satisfying

$$\phi_f^2(\omega) = \omega \text{ and } u_\omega^f \circ u_{\phi_f(\omega)}^f = I, \quad \forall\ \omega\ \in\ \Omega$$

such that

$$Pf(\omega) = \frac{1}{2}[f(\omega) + u_\omega^f(f(\phi_f(\omega)))].$$

Therefore, for each $f \in C(\Omega, X)$, we get $(2P - I)f(\omega) = u_\omega^f(f(\phi_f(\omega)))$. This implies that $2P - I \in \overline{\mathcal{G}^2(C(\Omega, X))}^a$. The conclusion follows from Corollary 5.1.2. □

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