Indian Institute of Information Technology Allahabad
Univariate and Multivariate Calculus
C3 End Semester Examination - Tentative Marking Scheme

Program: B.Tech. 2^{nd} Semester
Duration: **2 hours** | Full Marks: 55
Date: June 27, 2023 | Time: 9:30 AM - 11:30 AM

Important Instructions:

1. Answer all questions. Writing on question paper is not allowed.
2. Attempt all the parts of questions 1 at the same place. Parts done separately will not be graded.
3. Number the pages of your answer booklet. On the back of the front page of your answer booklet, make a table (as shown below) to indicate the page number in which respective questions have been answered. If you did not attempt a particular question, write down NA.

Question No.	1	2	3	4	5	6	7
Page No.							

4. Use of any electronic gadgets is not allowed.
5. This question paper contains two pages.

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1. Mention whether the following statements are true or false. In either case, provide proper justification. [3+2+2+2]
   (a) Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that $f(x) \geq 0$ for all $x \in [0,1]$. If $\int_0^1 f(x)dx = 0$, then $f(x) = 0$ for all $x \in [0,1]$.
   (b) The sequence $\left(\frac{\sin n}{n}, \frac{\pi^n}{n!}\right)$ is convergent.
   (c) The set $\{(x,y,z) \in \mathbb{R}^3 : z = 0\}$ is open in $\mathbb{R}^3$.
   (d) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y) = \sqrt{|xy|}$. Then the directional derivatives of f at $(0,0)$ exist in all directions.

2. Determine the values of α for which the integral

$$\int_0^\infty x^{\alpha-1}e^{-x}\,dx$$

converges. [6]

Solution: $\int_0^\infty x^{\alpha-1}e^{-x}\,dx = \int_0^1 x^{\alpha-1}e^{-x}\,dx + \int_1^\infty x^{\alpha-1}e^{-x}\,dx.$ [1]

Let $g_1(x) = \frac{1}{x^{1-\alpha}}$ and $g_2(x) = \frac{1}{x^2}$.

Since $\lim_{x\to 0} \frac{x^{\alpha-1}e^{-x}}{g_1(x)} = 1$, by LCT $\int\limits_0^1 x^{\alpha-1}e^{-x}\, dx$ converges for $1-\alpha<1$, that is, $\alpha>0$. [2]

Since $\lim_{x\to\infty} \frac{x^{\alpha-1}e^{-x}}{g_2(x)} = 0$, by LCT $\int\limits_0^1 x^{\alpha-1}e^{-x}\, dx$ converges for all $\alpha \in \mathbb{R}$. [2]

Hence, $\int\limits_0^\infty x^{\alpha-1}e^{-x}\, dx$ converges if and only if $\alpha>0$. [1]

3. Find the length of the curve

$$y = \frac{x^3}{12} + \frac{1}{x}, \quad 1 \le x \le 4.$$ [4]

Solution: The length of the curve is $L = \int_a^b \sqrt{1+[f'(x)]^2}dx$. [1]

Hence, the length of the graph over [1,4] is $L = \int_1^4 \sqrt{1+\left[\frac{x^2}{4} - \frac{1}{x^2}\right]^2}dx = \int_1^4 \left(\frac{x^2}{4}+\frac{1}{x^2}\right)dx = 6$. [1+1+1]

4. The region bounded by the curve $y = x^2+1$ and the line $y=-x+3$ is revolved about the x-axis to generate a solid. Find the volume of the solid using the Washer method. [6]

Solution: Here, outer radius is $R(x) = -x+3$ and inner radius is $r(x) = x^2+1$. [1+1]

The volume is given by $V = \int_a^b \pi([R(x)]^2 - [r(x)]^2)dx$. [1]

Hence, $V = \int_{-2}^1 \pi((-x+3)^2 - (x^2+1)^2)dx = \pi\int_{-2}^1 (8-6x-x^2-x^4)dx = \frac{117\pi}{5}$. [1+1+1]

5. Consider the curve $r = 2\cos 2\theta$. [6]

(a) Sketch the graph of the above curve.

(b) Does the point $(2, \frac{\pi}{2})$ lie on this curve.

(c) Find the area of the region in the plane enclosed by the curve.

Solution:

To sketch the graph we take some points

θ	0	$\pi/6$	$\pi/4$	$\pi/2$
r	2	1	0	-2

The area of half petal is given by $A = \int_a^b \frac{1}{2}r^2 d\theta = \int_0^{\pi/4} \frac{1}{2}(2\cos 2\theta)^2 d\theta = \pi/4$. [1+1+1]

Therefore the required area under curve is $8 \times \pi/4 = 2\pi$. [1]

Here, the point $(2,\pi/2)$ lies on the curve because $(2,\pi/2) = (-2,-\pi/2)$. [1]

Sketch [1]

6. Find the absolute maxima and absolute minima of the function

$$f(x,y) = 2+2x+4y-x^2-y^2$$

on the triangular region in the first quadrant bounded by the lines $x=0, y=0$ and $y=9-x$. [12]

Solution: $\nabla f(x,y) = (2-2x, 4-2y)$

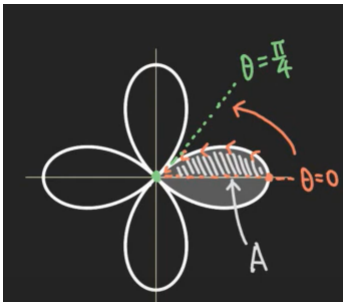

Figure 1: Sketch

The critical point is $(1,2)$ and $f(1,2) = 7$. [1]

On boundary $y = 0$, $g(x) = f(x,0) = 2 + 2x - x^2$ and $g'(x) = 2 - 2x$. [1]

Hence critical point in the interior of the boundary is $(1,0)$ and $f(1,0) = 3$. [1]

On boundary $x = 0$, $g(y) = f(0,y) = 2 + 4y - y^2$ and $g'(y) = 4 - 2y$. [1]

Hence critical point in the interior of the boundary is $(0,2)$ and $f(0,2) = 6$. [1]

On boundary $y = 9 - x$, $g(x) = f(x, 9-x) = -43 + 16x - 2x^2$ and $g'(x) = 16 - 4x$. [1]

Hence critical point in the interior of the boundary is $(4,5)$ and $f(4,5) = -11$. [1]

The corner points are $(0,0), (9,0)$ and $(0,9)$ with $f(0,0) = 2$, $f(9,0) = -61$ and $f(0,9) = -43$. [1+1+1]

Thus, the absolute maximum value is 7 which is attained at $(1,2)$ and absolute minimum value is -61 which is attained at $(9,0)$. [1+1]

7. Let D denote the solid enclosed by the spheres $x^2 + y^2 + (z-1)^2 = 1$ and $x^2 + y^2 + z^2 = 3$. Find the volume of D using spherical coordinates. [12]

Solution.

The volume is given by

$$\int_0^{2\pi} \int_0^{\pi/6} \int_0^{\sqrt{3}} \rho^2 \sin\phi d\rho d\phi d\theta \ [1+2+2+1] + \int_0^{2\pi} \int_{\pi/6}^{\pi/2} \int_0^{2\cos\phi} \rho^2 \sin\phi d\rho d\phi d\theta \ [2+2]$$