L'HÔPITAL'S RULES

1. Indeterminate Forms

In this lecture we will discuss limit theorems that involve cases which cannot be determined by previous limit theorems.

Let $f, g : [a, b] \to \mathbb{R}$ be two functions such that $g(x) \neq 0$ for all $x \in [a, b]$, $x \neq c$, and $c \in [a, b]$. We have seen that if $\lim_{x\to c} f(x) = A$ and $\lim_{x\to c} g(x) = B$, and if $B \neq 0$, then $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{A}{B}$. If $B = 0$, then no conclusion was deducted. Consider the following two situations:

(1) If $B = 0$, and $\lim_{x\to c} \frac{f(x)}{g(x)}$ exists, then $A = \lim_{x\to c} f(x) = \lim_{x\to c} \frac{f(x)}{g(x)} g(x) = 0$.
(2) If $g(x) > 0$ for all $x \in [a, b]$, $x \neq c$, $A > 0$ and $B = 0$, then $\lim_{x\to c} \frac{f(x)}{g(x)} = \infty$ (Try to prove this!). Similarly, if $A < 0$ and $B = 0$, then $\lim_{x\to c} \frac{f(x)}{g(x)} = -\infty$.

The case $A = B = 0$ is still remaining. In this case, the limit of the quotient $\frac{f}{g}$ is said to be "indeterminate", and depending on the functions f and g the limit may not exist or may be any real number. The symbolism $\frac{0}{0}$[1] is used to refer to this situation. For instance, if $\lambda \in \mathbb{R}$, then for $f(x) = \lambda x$ and $g(x) = x$,

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{\lambda x}{x} = \lambda.$$

Thus, the indeterminate form $\frac{0}{0}$ can lead to any real number λ as a limit.

Other indeterminate forms are represented by the symbols $\frac{\infty}{\infty}$, $0 \cdot \infty$, 0^0, 1^∞, ∞^0 and $\infty - \infty$. These notations correspond to the indicated limiting behaviour of the functions f and g. We will focus on the indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$. Other indeterminate cases are usually reduced to the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by taking logarithms, exponentials, or algebraic manipulations.

2. L'Hôpital's Rules

Most calculus books discuss L'Hôpital's Rules for $\frac{0}{0}$ and $\frac{\infty}{\infty}$ indeterminate forms separately. In the next theorem, we will give a general statement of L'Hôpital's Rule which covers the above two forms and some other cases as well.

[1] The limit theorem for the $\frac{0}{0}$ case, later known as L'Hôpital's (pronounced as Lowpeetal) Rule, was actually discovered by Bernoulli. It appeared in the first textbook on differential calculus book written by L'Hôpital's in 1696. Bernoulli was L'Hôpital's teacher.

Theorem 1 (L'Hôpital's Rule)**.** *Let J be an open interval. Let either $a \in J$ or a is an endpoint of J (It is possible that $a = \pm\infty$). Assume that*

(1) $f, g : J \setminus \{a\} \to \mathbb{R}$ is differentiable,
(2) $g(x) \neq 0 \neq g'(x)$ for $x \in J \setminus \{a\}$,
(3) $A = \lim_{x\to a} f(x) = \lim_{x\to a} g(x)$, where A is either 0 or ∞, and
(4) $B = \lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists either in $\mathbb{R}$ or $B = \pm\infty$.

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = B.$$

Proof. We apply Cauchy mean value theorem to prove the case where $A = 0$, $a \in \mathbb{R}$, and $B \in \mathbb{R}$. Interested readers may refer to [1] for the proof of other cases.

Define $f(a) = 0 = g(a)$ to make f and g continuous on J. Let (x_n) be a sequence in J such that $x_n > a$ or $x_n < a$ for all n and $x_n \to a$. By CMVT, $\exists\ c_n$ between a and x_n such that

$$\frac{f(x_n)}{g(x_n)} = \frac{f(x_n) - f(a)}{g(x_n) - g(a)} = \frac{f'(c_n)}{g'(c_n)}.$$

Now, $x_n \to a$ implies that $c_n \to a$. By hypothesis, the sequence $\frac{f'(c_n)}{g'(c_n)} \to B$. Thus, $\frac{f(x_n)}{g(x_n)} \to B$, and hence $\lim_{x\to a} \frac{f(x)}{g(x)} = B$. □

Before looking at some examples, note that when a is an endpoint of J, we get the statement of L'Hôpital's Rule for left and right hand limits.

Examples

(1) $\lim_{x\to 2} \frac{\sqrt{x^2+5}-3}{x^2-4} = \lim_{x\to 2} \frac{\frac{x}{\sqrt{x^2+5}}}{2x} = \frac{\frac{2}{3}}{4} = \frac{1}{6}$.
(2) $\lim_{x\to\infty} (x^3 + 4x^2 + 13x + 1)^{1/3} - x = \lim_{y\to 0^+} \frac{(1+4y+13y^2+y^3)^{1/3}-1}{y} = \lim_{y\to 0^+} \frac{1}{3}(1 + 4y + 13y^2 + y^3)^{-2/3}(4 + 26y + 3y^2) = \frac{4}{3}$.

Examples

(1) $\lim_{x\to\infty} \frac{x^2+2x+3}{3x^2+2x+1} = \lim_{x\to\infty} \frac{2x+2}{6x+2} = \lim_{x\to\infty} \frac{2}{6} = \frac{1}{3}$.
(2) $\lim_{x\to\infty} \frac{x^3}{x^2-1} - \frac{x^3}{x^2+1} = \lim_{x\to\infty} \frac{2x^3}{x^4-1} = \lim_{x\to\infty} \frac{6x^2}{4x^3} = \lim_{x\to\infty} \frac{3}{2x} = 0$.

Other Indeterminate Forms.

(1) $\lim_{x\to 0^+} x \log x = \lim_{x\to 0^+} \frac{\log x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = 0$.
(2) $\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x}\right) = \lim_{x\to 0^+} \frac{\sin x - x}{x \sin x} = \lim_{x\to 0^+} \frac{\cos x - 1}{\sin x + x\cos x} = \lim_{x\to 0^+} \frac{-\sin x}{2\cos x - x \sin x} = 0$.
(3) $\lim_{x\to\infty} (1 + \frac{1}{x})^x = \lim_{x\to\infty} e^{x \log(1+\frac{1}{x})}$. We note that $\lim_{x\to\infty} x \log(1+\frac{1}{x}) = \lim_{x\to\infty} \frac{\log(1+\frac{1}{x})}{\frac{1}{x}} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = 1$. Thus, $\lim_{x\to\infty}(1+\frac{1}{x})^x = e$.

References

[1] Tom M. Apostol, Calculus. Vol. I: One-variable calculus, with an introduction to linear algebra, Second edition, 1967.