MEAN VALUE THEOREM

1. Tangents and Rate of Change

There are two important ways of looking at derivatives. One interpretation is physical and the other is geometric.

Let $f : [a, b] \to \mathbb{R}$ be a differentiable function. We can interpret the difference quotient $\frac{f(x+h)-f(x)}{h}$ as the *average rate of change* of f over the interval from x to $x + h$. The *instantaneous rate of change* of f with respect to x at x_0 is the derivative $f'(x_0) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$. Thus, instantaneous rates are limits of average rates.

Problem 1. *The area A of a circle is related to its diameter D by the equation $A = \frac{\pi}{4}D^2$. How fast does the area change w.r.t the diameter when the diameter is* 10 *m.*

Solution. The rate of change of the area w.r.t the diameter is $\frac{dA}{dD} = \frac{\pi D}{2}$. When $D = 10$ m, the area changes w.r.t the diameter at the rate of 5π m. $\square$

If we think of the domain as the time interval and $f(x_0 + h) - f(x_0)$ as the distance travelled by a particle in h units of time, then the velocity is $\frac{f(x_0+h)-f(x_0)}{h}$. The derivative which is the limit of these velocities as $h \to 0$ is called the instantaneous velocity of the motion of the particle at the instant $x = x_0$.

The geometric interpretation of the derivative $f'(x)$ is that it is the slope of the tangent line at $(x, f(x))$ to the graph $\{(x, f(x)) : x \in [a, b]\}$.

Problem 2.
(1) *Find the slope of the curve $y = \frac{1}{x}$ at any point $x = a \neq 0$. What is the slope of at $x = -1$.*
(2) *At which point(s), the slope equals $-\frac{1}{4}$.*
(3) *What happens to the tangent to the curve at the point $(a, \frac{1}{a})$ as a changes.*

Solution.
(1) Here, $f(x) = \frac{1}{x}$. Then $f'(x) = -\frac{1}{x^2}$. Hence, the slope at $(a, \frac{1}{a})$ is $-\frac{1}{a^2}$.
(2) $-\frac{1}{a^2} = -\frac{1}{4} \Rightarrow a = 2$ or $a = -2$. Thus, the curve has slope $-\frac{1}{4}$ at the points $(2, \frac{1}{2})$ and $(-2, -\frac{1}{2})$.
(3) The slope $-\frac{1}{a^2}$ is always negative if $a \neq 0$. As $a \to 0^+$ (or 0^-), the slope approaches $-\infty$ and the tangent becomes increasingly steep. As a moves away from 0 in either direction, the slope approaches 0 and the tangent levels off to become horizontal.

$\square$

2. Mean Value Theorem

Definition 3 (Local Exremum)**.** *(1) Let J be an interval and $f : J \to \mathbb{R}$ be a function. We say that a point $c \in J$ is a point of local maximum if there exists $\delta > 0$ such that $(c - \delta, c + \delta) \subset J$ and $f(x) \le f(c)$ for all $x \in (c - \delta, c + \delta)$. A local minimum is defined similarly.*

(2) A point c is said to be a local extremum if it is either a local maximum or a local minimum.

(3) A point x_0 is called a point of global maximum on J if $f(x) \le f(x_0)$ for all $x \in J$. Global minimum and then global extremum are defined similarly.

Examples

(1) Let $f : [a, b] \to \mathbb{R}$ be defined as $f(x) = x$. Then b is a point of global maximum but not a local maximum. Also, a is a point of global minimum but not a local minimum.

(2) Consider $f : [-2\pi, 2\pi] \to \mathbb{R}$ where $f(x) = \cos x$. The point $x = 0$ is a local maximum as well as a global maximum. What do you think about the points $x = \pm 2\pi$.

Proposition 4. *Let $f : J \to \mathbb{R}$ be differentiable at $c \in J$. If f has local extremum at c, then $f'(c) = 0$.*

Proof. Suppose f has a local maximum at c. Then $\exists\ \delta > 0$ such that $f(x) \le f(c)$ for all $x \in (c - \delta, c + \delta)$. In other words, for all $h \in (-\delta, \delta)$, we have $f(c + h) \le f(c)$ and $f(c - h) \le f(c)$. Since f is differentiable at c,

$$f'(c) = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \le 0 \text{ and } f'(c) = \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \ge 0.$$

This implies that $f'(c) = 0$. □

Proposition 5 (Rolle's Theorem)**.** *If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b) and if $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.*

Proof. Exercise. □

Geometrically, Rolle's Theorem says that there exists a point $c \in (a, b)$ such that the tangent at $(c, f(c)$ is parallel to x-axis.

Rolle's Theorem together with IVP are used to check the existence and uniqueness of roots of continuous functions in certain intervals as illustrated in the following examples.

Examples

(1) Let $f(x) = x^3 + px + q$ for $x \in \mathbb{R}$, where $p, q \in \mathbb{R}$, $P > 0$. We observe that $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$. Hence by IVP, there $a \in \mathbb{R}$ such that $f(a) = 0$. Thus, f has at least one real root. Suppose there is $b \in \mathbb{R}$ such that $f(b) = 0$. Rolle's Theorem implies the existence of $c \in (a, b)$ such that $f'(c) = 0$. But $f'(x) = 3x^2 + p \neq 0$ for any $x \in \mathbb{R}$ since $p > 0$.

(2) If $f(x) = x^4 + 2x^3 - 2$ for $x \in \mathbb{R}$, then $f(0) = -2 < 0$ and $f(1) = 1 > 0$. Therefore, by IVP f will have at least one root in $[0, 1]$. Moreover, $f'(x) = 4x^3 + 6x^2 > 0$ for $x \in (0, \infty)$. So, f has at most one root in $[0, \infty)$. This implies that f has a unique root in $[0, \infty)$.

Now we state the most important result in differentiation.

Theorem 6 (Mean Value Theorem (MVT))**.** *If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that*

$$f(b) - f(a) = f'(c)(b - a).$$

Proof. Define $F : [a, b] \to \mathbb{R}$ by

$$F(x) = f(x) - f(a) - \left(\frac{f(b) - f(a)}{b - a}\right)(x - a).$$

Now, apply Rolle's Theorem to the function F. □

Remark 7. *(1) The mean value theorem (in short, MVT) is also known as Lagrange's mean value theorem. MVT is crucial in characterizing constant functions, monotonic functions, and convex/concave functions. Such characterizations can only be obtained using MVT.*

(2) If we write $b = a + h$, then MVT could be stated as follows:

$$f(a + h) = f(a) + hf'(a + \theta h) \text{ for some } \theta \in (0, 1). \text{ (Verify!)}$$

2.1. **Applications of MVT.**

(1) Let $f : [a, b] \to \mathbb{R}$ be differentiable such that $f'(x) = 0$ for all $x \in [a, b]$. Then f is constant.

Proof. For any $x, y \in [a, b]$, by MVT, we have $f(y) - f(x) = f'(z)(y - x)$ for some z between x and y. Since $f'(z) = 0$, we get $f(x) = f(y)$ for all $x, y \in [a, b]$. This implies that f is a constant function. (Try to prove this ab initio!) □

(2) Let $f : [a, b] \to \mathbb{R}$ be differentiable. If $f'(x) \geq 0$ (respectively, $f'(x) > 0$) for all $x \in [a, b]$, then f is increasing (respectively, strictly increasing). We have a similar result for decreasing functions.

Proof. For $x, y \in [a, b]$ such that $x < y$, we have by MVT, $f(y) - f(x) = f'(z)(y - x)$ for some z between x and y. Since $f'(z) \geq 0$ (respectively, $f'(z) > 0$) and $y - x > 0$, we get $f(y) \geq f(x)$ (respectively, $f(y) > f(x)$). Thus, f is increasing (respectively, increasing). □

(3) MVT is quite useful in proving certain inequalities. For example, can you find out which is greater, e^π or π^e. Let us prove a more general inequality.
■ If $e \leq a < b$, then $a^b > b^a$.

Proof. Let $0 < x < y$ and $f(x) = \log x$ on $[x, y]$. Using MVT, try to prove

$$\frac{y - x}{y} < \log \frac{y}{x} < \frac{y - x}{x}. \tag{1}$$

Now, using (1), we have

$$\frac{b - a}{b} < \log \frac{b}{a} < \frac{b - a}{a}.$$

Since $a \log \frac{b}{a} < b - a$, we have $\frac{b^a}{a^a} = e^{a \log \frac{b}{a}} < e^{b-a}$. That is, $b^a < e^{b-a} a^a$. If $e \leq a$, then $e^t \leq a^t$ for $t \geq 0$ (Why?). Thus, we conclude that

$$b^a < e^{b-a} a^a < a^{b-a} a^a = a^b.$$

□

Theorem 8 (Cauchy's Mean Value Theorem (CMVT))**.** *If $f, g : [a, b] \to \mathbb{R}$ are continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof. Exercise. □