DIFFERENTIABILITY

The basic idea of differential calculus is to study the local behaviour of a function at a point by its first order (linear) approximation at the same point.

Throughout J will denote an interval, and $c \in J$. Let $f : J \to \mathbb{R}$ be a function. We want to approximate $f(x)$ for x near c. If $E(x) = f(x) - a - b(x - c)$ is the error by taking the value of $f(x)$ as $a + b(x-c)$ near c, we want the error to go to zero much faster than x goes to c. That is, $\lim_{x\to c} \frac{f(x)-a-b(x-c)}{x-c} = 0$. If this is true, then $\lim_{x\to c}(f(x) - a - b(x-c)) = 0$ (Why?). This implies that $\lim_{x\to c} f(x) = a$. If f is continuous at c, then $f(c) = a$. Thus, we can approximate f at c if there exists a real number b such that $\lim_{x\to c} \frac{f(x)-f(c))}{x-c} = b$. If this happens, we say that f is differentiable at c, and denote the real number b by $f'(c)$.

Definition 1. *Let $f : J \to \mathbb{R}$. Then f is said to be differentiable at c if $\exists\ b \in \mathbb{R}$ such that $\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = b$. In this case, the value of the limit, that is, b is denoted by $f'(c)$ and is called the derivative of f at c.*

If f is differentiable at every point of J, we say that f is differentiable on J. In such a case, we obtain a new function from J to $\mathbb{R}$ given by $c \mapsto f'(c)$. This function is denoted by f' and is called the derivative function of f. We sometime denote f' by $\frac{df}{dx}$ or $\frac{dy}{dx}$ when $y = f(x)$. Likewise, $f'(c)$ is often denoted by $\frac{df}{dx}|_{x=c}$ or $\frac{dy}{dx}|_{x=c}$.

Remark 2. *(1) In $\epsilon - \delta$ form, we say that f is differentiable at c if for every $\epsilon > 0$, $\exists$ a $\delta > 0$ such that*

$$x \in J,\ 0 < |x - c| < \delta \Rightarrow |f(x) - f(c) - b(x - c)| < \epsilon|x - c|.$$

(2) It is useful to use the variable h for the increment $x - c$. So, f is differentiable at c, if $\exists\ b \in \mathbb{R}$ such that $\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = b$.

Examples

(1) Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$. Then $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{|h|}{h}$. Let $x_n = \frac{1}{n}$ and $y_n = -\frac{1}{n}$. Then $x_n \to 0$ and $y_n \to 0$. But $\frac{|x_n|}{x_n} \to 1$ and $\frac{|y_n|}{y_n} \to -1$. Hence, $\lim_{h\to 0} \frac{|h|}{h}$ does not exist, and therefore, f is not differentiable at 0. On the other hand, verify that f is differentiable at each $c \in \mathbb{R}$, $c \neq 0$, and $f'(c) = 1$ if $c > 0$ and $f'(c) = -1$ if $c < 0$.

(2) Let $f(x) = x^2 \sin \frac{1}{x}$ when $x \neq 0$ and $f(0) = 0$. At the point 0, $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{h^2 \sin(1/h)}{h} = 0$ (Why?). Thus, f is differentiable at 0. It is clear that f

differentiable at other points as it is the product of two differentiable functions, and $f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$. Is the function f' differentiable at 0.

1. Properties of Derivatives

The following result gives a powerful, and a very simple characterization of differentiability of a function at a point

Proposition 3 (Carathéodory's Lemma)**.** *The function $f : J \to \mathbb{R}$ is differentiable at c iff $\exists$ a function $f_1 : J \to \mathbb{R}$ such that*

$$f(x) - f(c) = (x - c) f_1(x), \tag{1}$$

and f_1 is continuous at c. In such a case, $f'(c) = f_1(c)$.

Proof. ($\Rightarrow$) Define $f_1 : J \to \mathbb{R}$ by

$$f_1(x) = \begin{cases} \frac{f(x)-f(c)}{x-c}, & \text{if } x \in J \setminus \{c\}, \\ f'(c), & \text{if } x = c. \end{cases}$$

Then f_1 satisfies the required properties.

($\Leftarrow$) Putting $x = c + h$ in Equation (1), we have

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0} f_1(c+h) = f_1(c).$$

Thus, f is differentiable at c and $f'(c) = f_1(c)$. □

The function f_1 is called an increment function associated with f and c.

Corollary 4. *If $f : J \to \mathbb{R}$ is differentiable at $c \in J$, then f is continuous at c.*

Theorem 5 (Algebra of Differentiable Functions)**.** *Let $f, g : J \to \mathbb{R}$ be differentiable at $c \in J$. Then the following hold:*

(1) $f + g$ is differentiable at c with $(f + g)'(c) = f'(c) + g(c)$.
(2) αf is differentiable at c with $(\alpha f)'(c) = \alpha f'(c)$ for any $\alpha \in \mathbb{R}$.
(3) fg is differentiable at c with $(fg)'(c) = f(c)g'(c) + f'(c)g(c)$.
(4) If f is differentiable at c with $f'(c) \neq 0$, then $\frac{1}{f}$ is differentiable at c and $(\frac{1}{f})'(c) = -\frac{f'(c)}{f(c)^2}$.

Corollary 6. *Let $D(J)$ (respectively $C(J)$) denotes the set of differentiable (respectively continuous) functions on J. Then $D(J)$ is a vector subspace of $C(J)$.*

To find the derivative of a composite function $u = g(y)$ where $y = f(x)$, we use the Chain rule or Substitution rule. Roughly speaking, the Chain rule is $\frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx}$. Can you cancel out dy? Then there would be nothing to prove!

Theorem 7 (Chain Rule)**.** *Let $f : J \to \mathbb{R}$ and $g : J_1 \to \mathbb{R}$ be functions such that $f(J) \subset J_1$, an interval. If f is differentiable at c, and g is differentiable at $f(c)$ then $g \circ f$ is differentiable at c and $(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$.*

Proof. Let $f_1 : J \to \mathbb{R}$ be the increment function associated with f and c, and let $g_1 : J_1 \to \mathbb{R}$ be the increment function associated with g and $f(c)$. Then

$$f(x) - f(c) = (x - c)f_1(x) \ \forall\ x \in J,\ \ g(y) - g(f(c)) = (y - f(c))g_1(y) \ \forall\ y \in J_1.$$

Since $f(J) \subset J_1$, from the above equations we have

$$g(f(x)) - g(f(c)) = [f(x) - f(c)]g_1(f(x)) = (x - c)g_1(f(x))f_1(x) \ \forall\ x \in J.$$

We see that the function $(g_1 \circ f) \cdot f_1 : J \to \mathbb{R}$ is continuous at c (How?). Therefore, by Carathéodory's Lemma, $g \circ f$ is differentiable at c and $(g \circ f)'(c) = g_1(f(c))f_1(c) = g'(f(c))f'(c)$. □

Problem 8. *Let $f : (0, \infty) \to \mathbb{R}$ satisfy $f(xy) = f(x)+f(y)$ for all $x, y \in (0, \infty)$. Suppose f is differentiable at $x = 1$. Show that f is differentiable at every $x \in (0, \infty)$. Find out $f'(x)$.*

Solution. We first observe that $f(1) = 0$, $f(\frac{1}{x}) = -f(x)$ and $f(\frac{x}{y}) = f(x) - f(y)$. Now,

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{1}{h} f\left(\frac{x+h}{x}\right) = \lim_{k \to 0} \frac{f(1+k)}{kx} \\ &= \frac{1}{x} \lim_{k \to 0} \frac{f(1+k) - f(1)}{k} = \frac{1}{x} f'(1). \end{aligned}$$

□