PROPERTIES OF CONTINUOUS FUNCTIONS

In this lecture will study the relations between the continuity of a function and its several geometric properties. Throughout this lecture, J will denote an interval.

1. Continuity and Boundedness

Definition 1. *Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function.*

(1) f is said to be bounded above on D if $\exists\ \alpha \in \mathbb{R}$ such that $f(x) \leq \alpha$ for all $x \in D$. Any such α is called an upper bound for f.

(2) f is said to be bounded below on D if $\exists\ \beta \in \mathbb{R}$ such that $f(x) \geq \beta$ for all $x \in D$. Any such β is called a lower bound for f.

(3) f is said to be bounded on D if it is bounded above on D and also bounded below on D.

Remark 2. *(1) f is bounded if and only if $\exists\ \gamma \in \mathbb{R}$ such that $|f(x)| \leq \gamma$ for all $x \in D$. Such a γ is called a bound for the absolute value of f.*

(2) Geometrically, f is bounded above means that the graph of f lies below some horizontal line, while f is bounded below means that its graph lies above some horizontal line.

Examples

(1) Let $f, g : \mathbb{R} \to \mathbb{R}$ be two functions defined by $f(x) = -x^2$ and $g(x) = x^2$. Then f is bounded above and g is bounded below on $\mathbb{R}$. Neither of these functions is bounded on $\mathbb{R}$.

(2) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{x^2}{x^2+1}$ is bounded on $\mathbb{R}$. We can easily observe that $0 \leq f(x) < 1$. Moreover, the bounds 0 and 1 are optimal, i.e.,

$$\inf\{f(x) : x \in \mathbb{R}\} = 0 \text{ and } \sup\{f(x) : x \in \mathbb{R}\} = 1. \text{ (Prove!)}$$

(3) In the above example, we see that the infimum of f is attained, i.e., $\exists\ c\ (= 0) \in \mathbb{R}$ such that $\inf\{f(x) : x \in \mathbb{R}\} = f(c)$. What to you think about the supremum? Is it attained?

Definition 3. *Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function.*

(1) f attains its upper bound on D if $\exists\ c \in D$ such that $\sup\{f(x) : x \in \mathbb{R}\} = f(c)$.

(2) f attains its lower bound on D if $\exists\ d \in D$ such that $\inf\{f(x) : x \in \mathbb{R}\} = f(d)$.

(3) f attains its bounds on D if it attains its upper bound and lower bound on D.

A bounded function need not be continuous. For example, the Dirichlet's function defined in Lecture 5 is bounded but not continuous. Also, a continuous function need not be bounded. For instance, let $D_1 = [0, \infty)$ and $f_1(x) = x$ for $x \in D_1$, or $D_2 = (0, 1]$ and $f_2(x) = \frac{1}{x}$ for $x \in D_2$. It is clear that both f_1 and f_2 are continuous. The function f_1 is unbounded because its domain D_1 is unbounded. To understand why f_2 is unbounded, we need the following concept.

Definition 4. *Let $D \subseteq \mathbb{R}$. We say that D is a closed set if*

$$(x_n) \text{ any sequence in } D \text{ and } x_n \to x \Rightarrow x \in D.$$

The interval $(0, 1]$ is not a closed set, since $(\frac{1}{n}) \in (0, 1]$ and $\frac{1}{n} \to 0$, but $0 \notin (0, 1]$. The interval $[a, b]$ is closed. To see this, consider any sequence (x_n) in $[a, b]$ such that $x_n \to x$. Since $a \leq x_n \leq b$ and $x_n \to x$, we have $a \leq x \leq b$ (Why?). Hence $[a, b]$ is a closed set.

Theorem 5. *Let D be a closed and bounded subset of $\mathbb{R}$, and $f : D \to \mathbb{R}$ be a continuous function. Then f is bounded and attains its bounds on D.*

Proof. Suppose f is not bounded on D. Then for every $n \in \mathbb{N}$, $\exists\ x_n \in D$ such that $|f(x_n)| > n$ (Why?). The sequence (x_n) is bounded since D is bounded. By the Bolzano-Weierstrass Theorem, (x_n) has a convergent subsequence (x_{n_k}). If $x_{n_k} \to x$, then $x \in D$ since D is closed. By the continuity of f, $f(x_{n_k}) \to f(x)$. Being convergent, the sequence $(f(x_{n_k}))$ is bounded. This contradicts the fact that $|f(x_{n_k})| > n_k$ for every $k \in \mathbb{N}$ and $n_k \to \infty$ as $k \to \infty$. Hence, f is bounded.

To show that f attains its bounds, let $m = \inf\{f(x) : x \in D\}$ and $M = \sup\{f(x) : x \in D\}$ (Why m and M exist?). There exists a sequence (x_n) in D such that $f(x_n) \to M$. Since D is bounded, the sequence is bounded and has a convergent subsequence, say (x_{n_k}). Let $x_{n_k} \to x$. Then $x \in D$ because D is closed. By the continuity of f, $f(x_{n_k}) \to f(x)$. This implies that $f(x) = M$ (Why?). Thus, f attains its upper bound. The proof for the lower bound case is similar. $\square$

2. Continuity and Monotonicity

Definition 6. *$f : J \to \mathbb{R}$ be a function.*

(1) f is monotonically increasing (strictly increasing) on J if

$$x, y \in J,\ x < y \Rightarrow f(x) \leq f(y) (f(x) < f(y)).$$

(2) f is monotonically decreasing (strictly decreasing) on J if

$$x, y \in J,\ x < y \Rightarrow f(x) \geq f(y) (f(x) > f(y)).$$

(3) *f is monotonic (strictly monotonic)) on J if f is either increasing (strictly increasing) or decreasing (strictly decreasing) on J.*

We can easily find an example of a function that is monotonic but not continuous. For example, $f(x) = [x]$ for $x \in \mathbb{R}$. (Exercise. Discuss the points of continuity of f). Similarly, a continuous function may not be monotonic, $f(x) = |x|$ for $x \in \mathbb{R}$.

Theorem 7. *Let $f : J \to \mathbb{R}$ be a function that is strictly monotonic on J. Then $f^{-1} : f(J) \to \mathbb{R}$ is continuous.*

Remark 8. (1) *If f is strictly monotonic on J, then f is one-one and its inverse $f^{-1} : f(J) \to \mathbb{R}$ is well defined (Verify!).*

(2) *In the above theorem, the function f need not be continuous, and the range of f need not be an interval.*

Exercise 9. *Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x + [x]$. Show that f is strictly increasing on $\mathbb{R}$ and f^{-1} is continuous on $f(\mathbb{R})$ even though f is not continuous at any $m \in \mathbb{Z}$.*

3. Continuity and Intermediate Value Property (IVP)

Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $f(a)$ and $f(b)$ are of opposite signs. If you draw the graphs of few such functions, you will see that the graph meets the x-axis.

Theorem 10 (Intermediate Value Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $f(a) < \lambda < f(b)$. Then $\exists\ c \in (a, b)$ such that $f(c) = \lambda$.*

3.1. **Some Applications of IVP.**

(1) Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $f(x) \neq 0$ for any $x \in [a, b]$. Then either $f > 0$ or $f < 0$.

(2) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous taking values in $\mathbb{Z}$ or in $\mathbb{Q}$. Then f is constant.

(3) Let $f : [a, b] \to \mathbb{R}$ be a non-constant continuous function. Then $f([a, b])$ is an interval. To see this, let $y_1, y_2 \in f([a, b])$. Assume $y_1 < y < y_2$. We need to show that $y \in f([a, b])$. Let $x_1, x_2 \in [a, b]$ such that $f(x_1) = y_1$, $f(x_2) = y_2$. By IVP, $\exists$ x between x_1 and x_2 such that $f(x) = y$.

Theorem 11. *Let $f : J \to \mathbb{R}$ be a one-one continuous function. Then f is strictly monotone.*