LIMITS

In lecture 2, we have seen the definition of limit of a sequence. In this lecture, we shall define the concept of a limit of a function at a point in $\mathbb{R}$. For this, we must assume that the domain of the function satisfies certain conditions. Let $J \subseteq \mathbb{R}$ and $a \in \mathbb{R}$ such that J contains $(a-r, a)$ and $(a, a+r)$ for some $r > 0$. That is, the domain J contains an open interval around a except possibly the point a itself. So, in the sequel, when we write $f : J \to \mathbb{R}$, we always mean that J satisfies the above condition.

Definition 1. *Let $f : J \to \mathbb{R}$ be a function. We say that $\lim_{x\to a} f(x)$ exists if there exists $\ell \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$x \in J \text{ and } 0 < |x - a| < \delta \Longrightarrow |f(x) - \ell| < \varepsilon.$$

If $\lim_{x\to a} f(x)$ exists, we say that the limit of the function f as x tends to a (or as $x \to a$) is ℓ and write $\lim_{x\to a} f(x) = \ell$.

Remark 2. *Note that a need not be in the domain of f. Even if $a \in J$, ℓ need not be $f(a)$, and if $\ell = f(a)$, this information is irrelevant to us.*

Proposition 3. *With the notation of Definition 1, the limit ℓ is unique.*

Proof. Exercise. □

Some properties of limits

The proof of the following theorem is similar to the proof of the Theorem 5 in lecture 5.

Theorem 4. $\lim_{x\to a} f(x) = \ell$ *iff for every sequence (x_n) in $J \setminus \{a\}$ with $x_n \to a$, we have $f(x_n) \to \ell$.*

Example 5. *Let $f(x) = \cos\frac{1}{x}$, where $x \neq 0$. We show that $\lim_{x\to 0} f(x)$ does not exist. Indeed, consider $x_n = \frac{1}{\pi n}$ for $n \in \mathbb{N}$. Then $x_n \to 0$ but $f(x_n) = (-1)^n$ does not converge. Are we done here? Convince yourself that we are!*

The following simple lemma is an analogue of the sandwich theorem for the limits of functions.

Lemma 6. *Let f, g, h be defined on J. Assume that $f(x) \leq h(x) \leq g(x)$ for all $x \in J$, and $\lim_{x\to a} f(x) = \ell = \lim_{x\to a} g(x)$. Then $\lim_{x\to a} h(x) = \ell$.*

Proof. Exercise. □

We now relate the concepts of continuity and limit.

Proposition 7. *Let $f : J \to \mathbb{R}$ be a function and $a \in J$. Then f is continuous at a iff $\lim_{x\to a} f(x)$ exists and is equal to $f(a)$.*

Theorem 8 (Algebra of Limits)**.** *Let $f, g : J \to \mathbb{R}$ be given and $\alpha \in \mathbb{R}$. Assume $\lim_{x\to a} f(x) = \ell$ and $\lim_{x\to a} g(x) = m$. Then*

(1) $\lim_{x\to a}(f+g)(x) = \ell + m$.

(2) $\lim_{x\to a}(\alpha f)(x) = \alpha\ell$.

(3) $\lim_{x\to a}(fg)(x) = \ell m$.

(4) If $\ell \neq 0$, then $f(x) \neq 0$ for all $x \in (a-\delta, a+\delta)$ for some $\delta > 0$, $x \neq a$. Moreover, $\lim_{x\to a}(\frac{1}{f})(x) = \frac{1}{\ell}$.

One-sided and Infinite Limits

We now consider one-sided limits. In order to define one-sided limits such as $\lim_{x\to a^+} f(x)$, we need to restrict x to those $x > a$. That is

Definition 9 (Left and right hand limits)**.**

- $\lim_{x\to a^+} f(x) = \ell$ *exists if for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$x > a,\ x \in J \text{ and } 0 < |x-a| < \delta \Longrightarrow |f(x) - \ell| < \varepsilon.$$

- $\lim_{x\to a^-} f(x) = \ell$ *exists if for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$x < a,\ x \in J \text{ and } 0 < |x-a| < \delta \Longrightarrow |f(x) - \ell| < \varepsilon.$$

The relationship between the one-sided limits and the limit $\lim_{x\to a} f(x)$ is given below.

Theorem 10. $\lim_{x\to a} f(x)$ *exists iff* $\lim_{x\to a^+} f(x)$ *and* $\lim_{x\to a^-} f(x)$ *exist and are equal.*

Definition 11 (Limits at infinity)**.**

- *Let $f : (a, \infty) \to \mathbb{R}$. We say that a limit of f as x tends to infinity exists*

  if there is a real number ℓ such that for every $\varepsilon > 0$, there exists $r \in \mathbb{R}$ such that for $x > r$ we have $|f(x) - \ell| < \varepsilon$.

  We then write $f(x) \to \ell$ as $x \to \infty$ or $\lim_{x\to\infty} f(x) = \ell$.

  In terms of sequences,

  $\lim_{x\to\infty} f(x) = \ell$ *iff for any sequence (x_n) in (a, ∞) such that $x_n \to \infty$, we have $f(x_n) \to \ell$.*

- *Let $f : (-\infty, a) \to \mathbb{R}$. We say that $\lim_{x\to-\infty} f(x) = \ell$ if for every $\varepsilon > 0$, there exists $s \in \mathbb{R}$ such that for $x < s$ we have $|f(x) - \ell| < \varepsilon$.*

Definition 12 (infinite limits).

- *Let* $f : J \to \mathbb{R}$. *We say that* $\lim_{x\to a} f(x) = \infty$ *if for every* $p > 0$, *there exists* $\delta > 0$ *such that* $x \in J$ *and* $0 < |x-a| < \delta \Longrightarrow f(x) > p$.
- *Let* $f : J \to \mathbb{R}$. *We say that* $\lim_{x\to a} f(x) = -\infty$ *if for every* $s < 0$, *there exists* $\delta > 0$ *such that* $x \in J$ *and* $0 < |x-a| < \delta \Longrightarrow f(x) < s$.

Exercise 13. *Formulate a definition for the statements*

$$\lim_{x\to\infty} f(x) = \infty \text{ and } \lim_{x\to 0^+} f(x) = -\infty.$$

Example 14. $\lim_{x\to 0^+} \frac{1}{x} = \infty$. *Let* $r > 0$. *Choose* $0 < \delta < \frac{1}{r}$. *Then for* $0 < x < \delta$, *we have* $f(x) = \frac{1}{x} > r$.

Asymptotes

The concept of a limit at $\pm\infty$ is useful in considering asymptotes of curves. Roughly speaking, a straight line is considered to be an asymptote of a curve if it comes arbitrarily close to that curve. We use this information to draw a graph of a curve.

Definition 15. *Let* $f : J \to \mathbb{R}$ *be a function.*

- *A straight line* $y = b$, $b \in \mathbb{R}$, *is called* **horizontal asymptote** *of the curve* $y = f(x)$ *if either*
$$\lim_{x\to\infty} f(x) = b \text{ or } \lim_{x\to-\infty} f(x) = b.$$
- *A straight line* $y = ax+b$, *where* $a, b \in \mathbb{R}$ *and* $a \neq 0$, *is called* **oblique asymptote** *of the curve* $y = f(x)$ *if either*
$$\lim_{x\to\infty} (f(x) - ax) = b \text{ or } \lim_{x\to-\infty} (f(x) - ax) = b.$$
- *A straight line* $x = c$, $c \in \mathbb{R}$, *is called* **vertical asymptote** *of the curve* $y = f(x)$ *if one or more of the following holds:*
$$f(x) \to \infty \text{ as } x \to c^-,\ f(x) \to -\infty \text{ as } x \to c^-,$$
$$f(x) \to \infty \text{ as } x \to c^+,\ f(x) \to -\infty \text{ as } x \to c^+.$$

Example. Let $f : (-\infty, 0) \cup (1, \infty) \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} \frac{2x-1}{x-1} & \text{if } x > 1, \\ \frac{3x^2+4x+1}{x} & \text{if } x < 0. \end{cases}$$

For $x > 1$, $f(x) = 2 + \frac{1}{x-1}$, and so $\lim_{x\to\infty} f(x) = 2$. Therefore, the straight line $y = 2$ is a horizontal asymptote of the curve $y = f(x)$.

Also, as $x \to 1^+$, $f(x) \to \infty$. Hence, the straight line $x = 1$ is a vertical asymptote of the curve $y = f(x)$.

For $x < 0$, $f(x) = 3x + 4 + \frac{1}{x}$. Thus, $\lim_{x\to\infty}(f(x) - 3x) = 4$. It follows that the $y = 3x + 4$ is an oblique asymptote of the curve $y = f(x)$. Moreover, $f(x) \to -\infty$ as $x \to 0^-$. Hence, the straight line given by $x = 0$ is a vertical asymptote of the curve $y = f(x)$. The graph of f is shown below.