CONTINUITY

In the previous lectures we studied real sequences, that is, real-valued functions defined on the subset $\mathbb{N}$ of $\mathbb{R}$. In this lecture we will consider real-valued functions whose domains are arbitrary subsets of $\mathbb{R}$.

Continuity of Functions

We first introduce the notion of continuity and see few examples.

Definition 1 ($\varepsilon - \delta$ Definition[1]). *Let $J \subset \mathbb{R}$. Let $f : J \to \mathbb{R}$ and $a \in J$. We say that f is continuous at a if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ whenever $x \in J$ and $|x - a| < \delta$.*

If f is continuous at each $a \in J$, then we say that f is continuous on J.

Remark 2. *The basic idea to show the continuity of f at a point a is to obtain an estimate of the form*

$$|f(x) - f(a)| \leq K_a |x - a|,$$

where K_a is a constant which may depend on a. This may not work always.

Examples.

- Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x$. We want to prove that f is continuous on $\mathbb{R}$.

  To check the continuity at any point $a \in \mathbb{R}$, we need to estimate $|f(x) - f(a)| = |x - a|$. If $\varepsilon > 0$ is given, we wish to find a $\delta > 0$ such that If $|x - a| < \delta \implies |f(x) - f(a)| = |x - a| < \varepsilon$. This suggests that we may take $\delta = \varepsilon$. Now, how do you write in the exam!!??

  Let $a \in \mathbb{R}$ and $\varepsilon > 0$. Let $\delta = \varepsilon$. Then for any x with $|x - a| < \delta$, we have $|f(x) - f(a)| = |x - a| < \delta = \varepsilon$. Thus, f is continuous at $a \in \mathbb{R}$. Since a is arbitrary, we conclude that f is continuous on $\mathbb{R}$.
- Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Let $a \in \mathbb{R}$ and $\varepsilon > 0$. We choose a $\delta < \min\{1, \frac{\varepsilon}{1+2|a|}\}$. For $x \in \mathbb{R}$ such that $|x - a| < \delta$, we have $|x - a| < 1$ so that

[1] Epsilon-delta proofs are first found in the works of Cauchy. The formal $\varepsilon - \delta$ definition of continuity is attributed to Bolzano and Weierstrass. For more details see [1].

$|x| \le |x-a| + |a| < 1 + |a|$. Now,

$$\begin{aligned} |f(x) - f(a)| = |x^2 - a^2| = |x+a||x-a| \\ &\le (|x| + |a|)|x-a| \\ &\le (1 + 2|a|)|x-a| \\ &< (1+2|a|)\delta < \varepsilon. \end{aligned}$$

Thus, f is continuous at a and hence on $\mathbb{R}$.

But now you might be wondering: how did we know in advance that such a δ will work. For this we have to go back to Remark 2. We want an estimate of the form $|f(x) - f(a)| \le K_a|x-a|$. So,

$$|f(x) - f(a)| = |x^2 - a^2| = |x+a||x-a| \le (|x| + |a|)|x-a|. \tag{1}$$

We need to estimate $|x|$ in terms of a. Suppose we have already found a δ that works. This implies that $|x| \le |x-a| + |a| < \delta + |a|$. We know that if a δ works, then any $\delta' \le \delta$ also works. Assume $\delta < 1$ (If we find a $\delta > 1$, we can choose the minimum of this δ and 1). It follows that $|x| < 1 + |a|$. Equation (1) now becomes

$$|f(x) - f(a)| \le (|x| + |a|)|x-a| \le (1+2|a|)|x-a|.$$

Therefore, if we make sure that $(1+2|a|)|x-a| < \varepsilon$, we get $|f(x) - f(a)| < \varepsilon$. In other words, we have to ensure that $|x-a| < \frac{\varepsilon}{1+2|a|}$. We also wanted $|x| < 1 + |a|$. Thus, we need to take $\delta < \min\{1, \frac{\varepsilon}{1+2|a|}\}$.

Exercise 3. *Using $\varepsilon - \delta$ argument, prove that the following functions are continuous.*

(1) For $a > 0$, let $f : [-a, a] \to \mathbb{R}$ be defined as $f(x) = x^2$.
(2) For $a > 0$, let $f : (a, \infty) \to \mathbb{R}$ be defined as $f(x) = \frac{1}{x}$.

Sequential Criterion for Continuity

We can characterize continuity of a function using the theory of sequence limits. The next result provides a very useful criterion to check continuity of a function at a point.

Theorem 4. *Let $a \in J$, and let $f : J \to \mathbb{R}$ be a function. Then f is continuous at a if and only if for every sequence (x_n) in J with $x_n \to a$, we have $f(x_n) \to f(a)$.*

Proof. Suppose f is continuous at a and $x_n \to a$. Let $\varepsilon > 0$. Then there exists a $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ whenever $|x-a| < \delta$. Since $x_n \to a$, for this $\delta > 0$, there exists a positive integer N such that $|x_n - a| < \delta$ for all $n \ge N$. This N serves our purpose. That is, for all $n \ge N$,

$$|x_n - a| < \delta \Longrightarrow |f(x_n) - f(a)| < \varepsilon \Longrightarrow f(x_n) \to f(a).$$

For the converse, let us assume that f is not continuous at a. Then there exists $\varepsilon > 0$ for which no $\delta > 0$ will satisfy the required $(\varepsilon - \delta)$ condition. For each $\delta = \frac{1}{n}$, $n \in \mathbb{N}$, there exist x_n such that $|x_n - a| < \frac{1}{n}$ with $|f(x_n) - f(a)| \geq \varepsilon$. This implies that $x_n \to a$ but $f(x_n) \nrightarrow f(a)$. This contradicts our hypothesis. Hence f is continuous at a. □

Example 5.
- *Let $f : \mathbb{R} \to \mathbb{R}$ be given by*

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{otherwise.} \end{cases}$$

This is known as Dirichlet's function. This function is not continuous at any point of $\mathbb{R}$.

Let $a \in \mathbb{Q}$. Choose a sequence of irrational numbers (x_n) converging to a. Then $f(x_n) = 0 \nrightarrow f(a) = 1$. Similarly, if $a \notin \mathbb{Q}$, we choose a sequence of rational numbers (x_n) converging to a. Then $f(x_n) = 1 \nrightarrow f(a) = 0$. So, f is not continuous at any $a \in \mathbb{R}$.

- *Let $f : [0, \infty) \to \mathbb{R}$ be given by*

$$f(x) = \begin{cases} 1, & \text{if } x = 0, \\ 1/q, & \text{if } x = p/q, \text{ where } p, q \in \mathbb{N} \text{ and } p, q \text{ have no common factors,} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

This function is known as Thomae's function. It is discontinuous at every rational in $[0, \infty]$. Let $a \in \mathbb{Q}$. Choose a sequence of irrational numbers (x_n) converging to a. Then $f(x_n) = 0 \nrightarrow f(a)$ as $f(a) \neq 0$. What happens at irrational numbers?

Problem 6. *Let $f : \mathbb{R} \to (0, \infty)$ be a function which satisfies $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. If f is continuous at 0, show that f is continuous at every $a \in \mathbb{R}$.*

Solution. Since $f(0) = f(0)^2$, we have $f(0) = 1$. Moreover, $f(x-x) = f(x)f(-x) \implies f(-x) = \frac{1}{f(x)}$. Let $a \in \mathbb{R}$ and $x_n \to a$. It follows that $x_n - a \to 0$. As f is continuous at 0, $f(x_n - a) \to f(0) = 1$. But $f(x_n - a) = f(x_n)f(-a) = \frac{f(x_n)}{f(a)} \to 1$. This implies that $f(x_n) \to f(a)$ (Why?). Hence f is continuous at a.

References

[1] Judith V. Grabiner, *The Origins of Cauchy's Rigorous Calculus*, United States, Dover Publications, 2012.