CAUCHY SEQUENCES AND SUBSEQUENCES

Cauchy Sequence

Let $x_n \to x$. Then for $\varepsilon > 0$, there exists a positive integer N such that $|x_n - x| < \varepsilon/2$ whenever $n \geq N$. Hence, for $n, m \geq N$ we have

$$|x_n - x_m| = |x_n - x + x - x_m| \leq |x_n - x| + |x_m - x| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This means that the terms of a converging sequence (x_n) get close (in sense of distance) to each other as m, n increases. Such a sequence is called a Cauchy sequence.

Definition 1 (Cauchy sequence)**.** *A sequence (x_n) is said to be Cauchy if for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$, we have $|x_n - x_m| < \varepsilon$.*

It is clear that any convergent sequence is Cauchy (How!). What about the converse? It turns out that the converse is also true. An easy proof of this fact uses the concept of subsequences, and a fundamental result known as *Bolzano-Weierstrass theorem.* But before everything let us see the following remark.

Remark 2. *If (x_n) is Cauchy, then $x_{n+1} - x_n \to 0$ as $n \to \infty$. The converse, however, does not hold. For example, if $x_n = \sqrt{n}$, then*

$$x_{n+1} - x_n = \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \to 0,$$

but (x_n) is not Cauchy. How do we prove that? Write in terms of quantifiers! Indeed, for $\varepsilon = 1$, and any $n_0 \in \mathbb{N}$, we have

$$|x_{4n_0} - x_{n_0}| = \sqrt{4n_0} - \sqrt{n_0} = \sqrt{n_0} \geq 1.$$

Lemma 3. *Any Cauchy sequence is bounded.*

Proof. Let (x_n) be Cauchy and $\varepsilon = 1$. Then there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$, we have $|x_n - x_m| < 1$. Take $m = N$. Then for $n \geq N$, $|x_n - x_N| < 1$. It follows that $|x_n| \leq |x_n - x_N| + |x_N| < 1 + |x_N|$ for all $n \geq N$. If we let $C = \max\{|x_1|, |x_2|, \ldots, |x_{N-1}|, |x_N|\}$, then $|x_n| \leq C$ for all $n \in \mathbb{N}$. □

The following result gives a useful sufficient condition for a sequence to be Cauchy.

Proposition 4. *Suppose $0 < \alpha < 1$ and (x_n) is a sequence satisfying the contractive condition:*

$$|x_{n+2} - x_{n+1}| \leq \alpha |x_{n+1} - x_n|, \quad \textit{for all } n \in \mathbb{N}.$$

Then (x_n) is a Cauchy sequence.

Example 5. *Let $x_1 = 1$ and $x_{n+1} = 1 + \frac{1}{x_n}$ for $n > 1$. We note that*

$$|x_{n+2}-x_{n+1}| = |1+\frac{1}{x_{n+1}}-1-\frac{1}{x_n}| = \left|\frac{x_{n+1}-x_n}{x_{n+1}x_n}\right| \text{ and } |x_{n+1}x_n| = \left|\left(1+\frac{1}{x_n}\right)x_n\right| = |x_n+1| \geq 2.$$

This implies that $|x_{n+2} - x_{n+1}| \leq \frac{1}{2}|x_{n+1} - x_n|$. Hence (x_n) is Cauchy.

Subsequences

Definition 6. *Let $f : \mathbb{N} \to \mathbb{R}$ be a sequence. Then a subsequence is the restriction of f to an infinite subset S of $\mathbb{N}$.*

Remark 7.
*(1) (**Well-Ordering Property**) Every nonempty subset S of $\mathbb{N}$ has a least element, i.e., there exists $\ell \in S$ such that $\ell \leq x$ for all $x \in S$.*
(2) Using item (1), one can prove that an infinite subset S of $\mathbb{N}$ can be listed as

$$\{n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots\}.$$

(3) In view of item (2), it is a standard practice to denote a subsequence of (x_n) as (x_{n_k}) where $n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$. Note that $n_k \to \infty$ as $k \to \infty$.
(4) **A very useful observation:** $n_k \geq k$ **for all** k.
(5) A subsequence is formed by deleting some of the terms of the sequence and retaining the remaining terms in the same order. For example, if (x_n) is a sequence, then $(x_1, x_3, \ldots, x_{2k-1}, \ldots)$ and $(x_2, x_4, \ldots, x_{2k}, \ldots)$ are subsequences of (x_n). In the former case, $n_k = 2k - 1$, and in the later, $n_k = 2k$ (k varies from 1 to ∞).
(6) $(-1, -1, \ldots, -1, \ldots)$ and $(1, 1, \ldots, 1, \ldots)$ are both subsequences of $((-1)^n)$. From this example, it is clear that a divergent sequence may have convergent subsequences. Does this happen all the time?
(7) $(\frac{1}{k^3})$ and $(\frac{1}{2^k})$ are subsequences of $(\frac{1}{n})$. What is n_k in both cases?

Lemma 8. *A sequence (x_n) converges to x if and only if every subsequence of (x_n) converges to x.*

Proof. Exercise. □

A remarkable fact about monotonic subsequences is the following.

Proposition 9. *Every sequence has a monotonic subsequence.*

Lemma 8 says that if a sequence is convergent, then all its subsequences converge to the same limit. What happens if a sequence is divergent? Does it have a convergent subsequence? The next theorem, which is called the Bolzano-Weierstrass Theorem,[1] answers this question.

Theorem 10. *Every bounded sequence has a convergent subsequence.*

Proof. Let (x_n) be a bounded sequence. By Proposition 9, (x_n) has a monotone subsequence (x_{n_k}). Since (x_n) is bounded, so is its subsequence (x_{n_k}). By Theorem 23 of Lecture 3, (x_{n_k}) is convergent. □

Now we state a necessary and sufficient condition for convergence.

Theorem 11 (Cauchy's criterion)**.** *A sequence is convergent if and only if it is Cauchy.*

Proof. ($\Longrightarrow$) Already done.

($\Longleftarrow$) Let (x_n) be a Cauchy sequence. By Lemma 3, (x_n) is bounded. Bolzano-Weierstrass Theorem implies the existence of a converging subsequence (x_{n_k}). Let $x_{n_k} \to x$. So, we have a candidate for the limit of (x_n). We claim that $x_n \to x$.

Let $\varepsilon > 0$. Since $x_{n_k} \to x$, there exists N_1 such that

$$|x_{n_k} - x| < \varepsilon/2, \quad \text{for all } k \geq N_1. \tag{1}$$

For the same ε, since (x_n) is Cauchy, there exists N_2 such that

$$|x_n - x_m| < \varepsilon/2, \quad \text{for all } n, m \geq N_2. \tag{2}$$

Let $N = \max\{N_1, N_2\}$. Fix an $k \geq N$, and let $m = n_k$. It follows from Equation (1) that

$|x_m - x| \leq \varepsilon/2$, because $m = n_k \geq k \geq N \geq N_1$ (Check this!).

On the other hand, from Equation (2) it follows that

$|x_n - x_m| < \varepsilon/2$, whenever $n \geq N$ (Why $m \geq N_2$!). Everything is ready now! For $n \geq N$, we have

$$|x_n - x| \leq |x_n - x_m| + |x_m - x| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This completes the proof. □

Cauchy's criterion is also referred as **Cauchy completeness of** $\mathbb{R}$.

[1]This theorem is considered as one of the most important tools in calculus. It is named after two great nineteenth-century mathematicians, Bernard Bolzano and Karl Weierstrass. These two mathematicians, the first German and the second Czech, rank with Cauchy among the founders of our subject.