SEQUENCES AND THEIR CONVERGENCE

Definitions and Examples

We start with the definition of a sequence.

Definition 1. *Let X be nonempty set. A sequence in X is a function $f : \mathbb{N} \to X$. We let $x_n = f(n)$ and call x_n the n-th term of the sequence. Generally we denote f by (x_n) or as an infinite tuple $(x_1, x_2, \ldots, x_n, \ldots)$. A sequence in $\mathbb{R}$ is called a real sequence. Likewise, a sequence in $\mathbb{C}$ is called a complex sequence.*

Throughout this course, unless stated otherwise, (x_n) will always denote a real sequence.

Examples. Some examples of sequences:

(1) Fix $c \in \mathbb{R}$ and define $x_n = c$ for all n. The sequence $(c, c, c, \ldots)$ is called a constant sequence.
(2) $(n) = (1, 2, 3, \ldots)$.
(3) $\left(\frac{1}{n}\right) = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right)$.
(4) $((-1)^n) = (-1, 1, -1, 1, \ldots)$.
(5) Let $x_1 = x_2 = 1$ and define $x_n = x_{n-1} + x_{n-2}$ for all $n > 2$. The sequence is $(1, 1, 2, 3, 5, 8, 13, \ldots)$. This sequence is called Fibonacci sequence[1].

The sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots$$

getting closer and closer to the number 0. We say that this sequence converges to 0 or that the limit of the sequence is the number 0. How should this idea be properly defined? The definition which we learned in school was something like this

A sequence x_n converges to a number ℓ if the terms of the sequence get closer and closer to ℓ.

The above definition is not precise. It is too vague, and sometimes misleading. What about the sequence

$$0.1, 0.01, 0.02, 0.001, 0.002, 0.0001, 0.0002, \ldots ?$$

This sequence should converge to 0 but the terms do not get steadily "closer and closer" but back off a bit at each second step. Also, the sequence

$$0.1, 0.11, 0.111, 0.1111, 0.1111, \ldots$$

is getting "closer and closer" to 0.2, but the sequence does not converge to 0.2. A smaller number ($\frac{1}{9}$, which it is also getting closer and closer to) is the correct limit.

[1] Also known as Fibonacci numbers. They appear in nature surprisingly often, for example, numbers of petals on a flower, number of spirals on a sunflower or a pineapple are typically Fibonacci numbers. The ratios of successive terms of the Fibonacci sequence tends to an irrational number called the golden ratio. The golden ratio appears in some patterns in nature, including the spiral arrangement of leaves.

Definition 2 (Limit of a Sequence)**.** *Let (x_n) be a real sequence. We say that (x_n) converges if there exists a real number ℓ such that for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that*

$$|x_n - \ell| < \varepsilon \textit{ for all } n \geq N.$$

In this case, (x_n) converges to ℓ. The number ℓ is called a limit of the sequence (x_n). We write $x_n \to \ell$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = \ell$. A sequence that is not convergent is said to be divergent.

The definition of convergence of (x_n) can be written in terms of quantifiers as follows:

$$\text{“}\exists\ \ell \in \mathbb{R}\ (\forall\ \varepsilon > 0\ (\exists\ N \in \mathbb{N}\ (\forall\ n \geq N\ (|x_n - \ell| < \varepsilon))))\text{”}.$$

Exercise 3. *What does it mean to say that a sequence (x_n) does not converge? Write it in words and then in terms of quantifiers.*

Examples. (Convergent sequences)

(1) If $x_n = c$ for all n, then $x_n \to c$. In fact, given $\varepsilon > 0$, we may take $N = 1$, so $|x_n - c| = 0$ for all $n \geq 1 = N$.

(2) If $x_n = \frac{1}{n}$ for $n \in \mathbb{N}$, then it is easy to see that the sequence should have limit $\ell = 0$. We claim that $x_n \to 0$. To see this, given any $\varepsilon > 0$, $|x_n - \ell| = |\frac{1}{n} - 0| = \frac{1}{n}$. We want to choose an N such that for all $n \geq N$, $|x_n - \ell| < \varepsilon$, i.e., $\frac{1}{n} < \varepsilon$ or $n > \frac{1}{\varepsilon}$. Such an n exists by the Archimedean property (How?).

Thus, choose an integer N such that $N > \frac{1}{\varepsilon}$ by the Archimedean property. Then for all $n \geq N$, we have

$$|x_n - \ell| = |\frac{1}{n} - 0| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

Hence, $\frac{1}{n} \to 0$.

(3) Consider $x_n = \frac{1}{2^n}$. If we look at the terms of the sequence, we can see that this sequence should converge to 0.

Let $\varepsilon > 0$. We have to find $N \in \mathbb{N}$ such that for all $n \geq N$

$$|x_n - \ell| = |\frac{1}{2^n} - 0| = \frac{1}{2^n} < \varepsilon.$$

Note that for all $n \in \mathbb{N}$, $2^n > n$ (Prove by induction!). Hence, $\frac{1}{2^n} < \frac{1}{n}$ for all $n \in \mathbb{N}$. Choose an integer N such that $N > \frac{1}{\varepsilon}$ by the AP, then for all $n \geq N$, we have

$$|x_n - \ell| = |\frac{1}{2^n} - 0| = \frac{1}{2^n} < \frac{1}{n} < \varepsilon.$$

Remark 4. *(1) If $|x_n - \ell| < \varepsilon$ for all $n \geq N$, then any $N_1 > N$ will also work. Thus, N is not unique.*

(2) From the the above examples we must have observed that the natural number N depends on the given $\varepsilon > 0$ while checking for convergence. That is why when we want to emphasize this, we sometime denote N by $N(\varepsilon)$.

Examples. (Divergent sequences)

(1) If $x_n = n$ for $n \in \mathbb{N}$, then it is not difficult to see intuitively that this (x_n) diverges. On the contrary, let $x_n \to \ell$. For $\varepsilon = 1$, $\exists\, N \in \mathbb{N}$ such that $|x_n - \ell| < 1$ for all $n \geq N$. In particular, for $n \geq N$, $x_n = n \in (\ell - 1, \ell + 1)$. This implies that $n < \ell + 1$ for all $n \in \mathbb{N}$, and hence $\mathbb{N}$ is bounded above. This is a contradiction. Therefore, (x_n) is divergent.

(2) If $x_n = (-1)^n$ for $n \in \mathbb{N}$, then (x_n) is divergent. Suppose $x_n \to \ell$. Choose $\varepsilon > 0$ such that $\varepsilon < 1$. Then there exists $N \in \mathbb{N}$ such that $x_n \in (\ell - \varepsilon, \ell + \varepsilon)$ for all $n \geq N$. In particular, $-1, 1 \in (\ell - \varepsilon, \ell + \varepsilon)$ $(x_{2N} = 1, x_{2N+1} = -1)$. Since, $1 < \ell + \varepsilon$ and $-1 > \ell - \varepsilon$, we have

$$2 < \ell + \varepsilon - (\ell - \varepsilon) = 2\varepsilon.$$

That is, $1 < \varepsilon$. This is a contradiction. Since ℓ is arbitrary, (x_n) is divergent.

Proposition 5 (Uniqueness of limit). *A convergent sequence has a unique limit.*

Proof. Exercise. ☐

Definition 6 (Bounded Sequence). *A sequence(x_n) is said to be bounded if there exists $C > 0$ such that $|x_n| \leq C$ for all $n \in \mathbb{N}$.*

In Example 2, the sequences in items $(1), (3)$ and (4) are bounded. The sequences in items (2) and (5) are unbounded.

Theorem 7 (Necessary condition for convergence). *A convergent sequence is bounded.*

Proof. Let $x_n \to \ell$ and $\varepsilon = 1$. There exists $N \in \mathbb{N}$ such that, for $n \geq N$, we have $|x_n - \ell| < 1$. This implies that

$$|x_n| \leq |x_n - \ell| + |\ell| < 1 + |\ell|, \text{ for } n \geq N.$$

If $C = \max\{|x_1|, |x_2|, \ldots, |x_{N-1}|, 1 + |\ell|\}$, then $|x_n| \leq C$, and hence, (x_n) is bounded. ☐

Is the converse of the above proposition true? Consider the sequence $((-1)^n)$. It is bounded but not convergent.

Limit Theorems

In general, verifying the convergence directly from the definition is a difficult task. In this section, we will study few results which will enable us to find limits of many sequences, and some sufficient conditions for the convergence of a sequence. Given two sequences (x_n) and (y_n), the product sequence, denoted by $(x_n y_n)$, is a new sequence (t_n) such that $t_n = x_n y_n$.

Theorem 8 (Algebra of Convergent Sequences). *Let $x_n \to x$, $y_n \to y$ and $\alpha \in \mathbb{R}$. Then*

(1) $x_n + y_n \to x + y$.

(2) $\alpha x_n \to \alpha x$.

(3) $x_n y_n \to xy$.

(4) If $x \neq 0$ and $x_n \neq 0$ for all n, then $\frac{1}{x_n} \to \frac{1}{x}$.

The following theorem is immediate now.

Theorem 9. *The set* c *of all convergent sequences of real numbers is a vector space over* $\mathbb{R}$. *Moreover, The map* $T : \mathrm{c} \to \mathbb{R}$ *defined by*

$$T((x_n)) = \lim_{n\to\infty} x_n$$

is a linear transformation.

Example 10. *Let* $x_n = \frac{1}{1^2+1} + \frac{1}{2^2+2} + \cdots + \frac{1}{n^2+n}$. *As* $\frac{1}{n^2+n} = \frac{1}{n} - \frac{1}{n+1}$, *we have*

$$x_n = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{n+1} \to 1.$$

An easy and a very useful result is the following

Theorem 11 (Sandwich Theorem)**.** *Let* (x_n), (y_n) *and* (z_n) *be sequences such that* $x_n \to \alpha$, $y_n \to \alpha$ *and* $x_n \le y_n \le z_n$ *for all* n. *Then* $z_n \to \alpha$.

Proof. Let $\varepsilon > 0$ be given. Since $x_n \to \alpha$ and $y_n \to \alpha$, there exist N_1 and N_2 such that

$$x_n \in (\alpha - \varepsilon, \alpha + \varepsilon) \text{ for all } n \ge N_1, \text{ and } y_n \in (\alpha - \varepsilon, \alpha + \varepsilon) \text{ for all } n \ge N_2.$$

If we let $N = \max\{N_1, N_2\}$, then for $n \ge N$, we have

$$\alpha - \varepsilon < x_n \le z_n \text{ and } z_n \le y_n < \alpha + \varepsilon.$$

That is, $z_n \in (\alpha - \varepsilon, \alpha + \varepsilon)$ for all $n \ge N$. It follows that $z_n \to \alpha$. $\square$

Examples.

- Let $x \in \mathbb{R}$. For each $n \in \mathbb{N}$, choose any element x_n such that $x - \frac{1}{n} < x_n < x + \frac{1}{n}$) (Why such x_n exists!). It follows from the algebra of convergent sequences and sandwich theorem that $x_n \to x$.
- We have $\frac{\sin n}{n} \to 0$, as $-\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}$.
- Let $x_n = \frac{1}{n^2+1} + \frac{2}{n^2+2} + \cdots + \frac{n}{n^2+n}$. Try to write down x_1, x_2 and x_3.
  Since $\frac{1}{n^2+n} \le \frac{1}{n^2+k} \le \frac{1}{n^2+1}$, $k = 1, 2, \ldots n$, we have

$$(1 + 2 + \cdots + n)\frac{1}{n^2+n} \le x_n \le (1 + 2 + \cdots + n)\frac{1}{n^2+1} \Rightarrow x_n \to \frac{1}{2}.$$

Some Important Limits

In this section, we will study some of the most often used sequences and their convergence.

Theorem 12. *(1) Let* $0 \le r < 1$. *Then* $r^n \to 0$.

(2) Let $|r| < 1$. *Then* $r^n \to 0$.

(3) Let $|r| < 1$. *Then* $nr^n \to 0$.

(4) Let $a > 0$. *Then* $a^{1/n} \to 1$.

(5) $n^{1/n} \to 1$.

(6) Fix $a \in \mathbb{R}$. *Then* $\frac{a^n}{n!} \to 0$.

Proof. (1) If $r = 0$, the result is obvious. If $0 < r < 1$, then $r = \frac{1}{h}$ for some $h > 0$. Using binomial theorem, we have

$$(1+h)^n = 1 + nh + \frac{n(n-1)}{2}h^2 + \cdots + h^n > nh,$$

since all terms are positive. This implies that $0 < r^n < \frac{1}{nh}$ for all n. By Sandwich theorem, $r^n \to 0$.

(2) Use part (1) and the fact that a sequence $x_n \to 0 \iff |x_n| \to 0$.

(3) It is enough to prove the result for $0 < r < 1$. Proceeding as in part (1),

$$(1+h)^n = 1 + nh + \frac{n(n-1)}{2}h^2 + \cdots + h^n > \frac{n(n-1)}{2}h^2,$$

since all terms are positive. Thus, $0 < nr^n < \frac{2}{h^2(n-1)}$ for all $n \geq 2$, and hence by Sandwich theorem, $nr^n \to 0$.

(4) If $a > 1$, write $a^{1/n} = 1 + h_n$, $h_n > 0$. This implies $a = (1+h_n)^n > nh_n$, and hence $0 < h_n < \frac{a}{n}$. This means $h_n \to 0$. Therefore, $a^{1/n} \to 1$ as desired.

If $0 < a < 1$, take $b = \frac{1}{a} > 1$.

(5) Assertions (5) and (6) are exercises.

□

Theorem 13. *Let $x_n \to x$. Let (s_n) be the sequence of arithmetic means defined by*

$$s_n = \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

Then $s_n \to x$.

Definition 14 (Sequences Diverging to $\pm\infty$)**.** *Let (x_n) be a sequence.*

- *We say that (x_n) diverges to $+\infty$ (or simply ∞), and write* $\lim_{n\to\infty} x_n = \infty$ *or $x_n \to \infty$ as $n \to \infty$,*

  if for any $r \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that $x_n > r$ whenever $n \geq N$.
- *Likewise, (x_n) diverges to $-\infty$, and write* $\lim_{n\to\infty} x_n = -\infty$ *or $x_n \to -\infty$ as $n \to \infty$,*

  if for any $s \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that $x_n < r$ whenever $n \geq N$.

Examples.

- Let $x_n = n$ and $y_n = 2^n$. Then both $x_n \to \infty$ and $y_n \to \infty$. (Prove!)
- Let $x_{2n-1} = 1$ and $x_{2n} = 2n$. This sequence is unbounded, and hence it is divergent (why!). However, it does not diverge to ∞. Try to write this in terms of quantifiers.

  If $r = 2$, then given any $N \in \mathbb{N}$, $2N - 1 \geq N$ and $x_{2N-1} < r$. Thus, $x_n \not\to \infty$.
- $(n!)^{1/n}$ diverges to ∞. To see this, let $r > 0$ be given. Since $\frac{r^n}{n!} \to 0$, for $\varepsilon = 1$, there exists $N \in \mathbb{N}$ such that $\frac{r^n}{n!} < 1$ whenever $n \geq N$. That is, $r^n < n!$ or $(n!)^{1/n} > r$ for $n \geq N$. Therefore, $(n!)^{1/n} \to \infty$.

The following result, called ratio test for sequences, is very useful.

Theorem 15 (Ratio Test)**.** *Let (x_n) be a sequence such that $x_n > 0$ for all n. Let* $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lambda$*. Then*

(1) If $\lambda < 1$, *then* $\lim_{n\to\infty} x_n = 0$.

(2) If $\lambda > 1$, *then* $\lim_{n\to\infty} x_n = \infty$.

Examples.

- Let $x_n = \frac{a^n}{n!}$, where $a \in \mathbb{R}$. Then $\frac{x_{n+1}}{x_n} = \frac{a}{n+1} \to 0$. It follows that $x_n \to 0$.
- Let $x_n = \frac{a^n}{n}$, where $a > 1$. Then $\frac{x_{n+1}}{x_n} = a$. It follows that $x_n \to \infty$.
- If $\lambda = 1$ in the previous theorem, the sequence (x_n) may converge or diverge. For example, $x_n = \frac{1}{n}$ and $x_n = n$.

Monotone Sequences

In this section we present a sufficient condition for the convergence of a sequence.

Definition 16. *We say that a sequence* (x_n) *is increasing (strictly increasing) if* $x_n \leq x_{n+1}$ $(x_n < x_{n+1})$ *for all* n, *and decreasing (strictly decreasing) if* $x_n \geq x_{n+1}$ $(x_n > x_{n+1})$ *for all* n. *A sequence* (x_n) *is said to be monotone if it is either increasing or decreasing.*

Note that any increasing sequence is bounded below by x_1, and a decreasing sequence is bounded above by x_1. Therefore, an increasing (decreasing) sequence is bounded if and only if it is bounded above (below).

Theorem 17 (Sufficient condition for convergence)**.** *(1) If a sequence* (x_n) *is increasing and bounded above, then it is convergent and*

$$\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}.$$

(2) If a sequence (x_n) *is decreasing and bounded below, then it is convergent and*

$$\lim_{n\to\infty} x_n = \inf\{x_n : n \in \mathbb{N}\}.$$

Example. Let $a > 0$ and $x_1 > 0$. Define $x_{n+1} = \frac{1}{2}(x_n + \frac{a}{x_n})$ for all $n \in \mathbb{N}$. We show that this sequence is bounded below and decreasing, hence convergent.

By AM-GM inequality,

$$x_{n+1} = \frac{x_n + \frac{a}{x_n}}{2} \geq \sqrt{a}. \text{ Further, } x_{n+1} - x_n = \frac{1}{2}\left(\frac{a - x_n^2}{x_n}\right) \leq 0.$$

Therefore, the sequence (x_n) is bounded below and decreasing. Can we find the limit?

Let $x_n \to \ell$. By the algebra of converging sequences, $\ell = \frac{1}{2}(\ell + \frac{a}{\ell})$ or $\ell^2 = a$. This means $\ell = \sqrt{a}$.

Problem 18. *Let* (x_n) *be bounded. Assume that* $x_{n+1} \geq x_n - 2^{-n}$. *Show that* (x_n) *is convergent.*

Solution. Since (x_n) be bounded, it is enough to show that sequence it is monotone. Let $y_n = x_n - \frac{1}{2^{n-1}}$. It is clear that (y_n) is bounded (How!). Moreover,

$$y_{n+1} - y_n = x_{n+1} - \frac{1}{2^n} - x_n + \frac{1}{2^{n-1}} \geq 0.$$

Hence, (y_n) is increasing. As (y_n) is bounded, it is convergent. This implies that (x_n) is convergent.