TRIPLE INTEGRALS; CHANGE OF VARIABLES IN TRIPLE INTEGRALS

In this lecture we shall extend the considerations of the last two lectures to functions defined on subsets of $\mathbb{R}^3$ (or more generally to bounded subsets of $\mathbb{R}^n$, where $n \geq 3$) and functions defined them in a straightforward manner.

1. Triple Integrals

Let $Q = [a, b] \times [c, d] \times [e, f]$ be a cuboid in $\mathbb{R}^3$. Every partition P of Q is of the form $P = P_1 \times P_2 \times P_3$ where P_1, P_2 and P_3 are partitions of $[a, b], [b, c]$ and $[e, f]$ respectively. For a given partition P and a bounded function f defined on Q, we can define $L(P, f), U(P, f)$, lower integral, upper integral and integral of f as we defined in the double integral case. If a function f on Q is integrable then the integral, called triple integral, is denoted by

$$\iiint_Q f(x, y, z)dxdydz \text{ or } \iiint_Q dV.$$

As we did in the double integral case, the definition of triple integral can be extended to any bounded regions in $\mathbb{R}^3$. One can also prove that every continuous function on Q is integrable.

Remark 1. *In the double integral case, the integral of positive function f is the volume of the region below the surface $z = f(x, y)$. In the triple integral case we do not have any such geometric interpretation, except the fact that $\iiint_D dxdydz$ is considered to be the volume of the region D.*

Let us now consider Fubini's Theorem for a function defined on bounded subsets of $\mathbb{R}^3$.

Theorem 2 (Fubini's Theorem)**.** *Let D be a bounded subset of $\mathbb{R}^3$ and let $f : D \to \mathbb{R}^3$ an integrable function. Suppose $D = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in R, f_1(x, y) \leq z \leq f_2(x, y)\}$, where $f_1, f_2 : R \to \mathbb{R}$ are integrable functions such that $f_1 \leq f_2$, and for each fixed $(x, y) \in R$, the integral $\int\limits_{f_1(x,y)}^{f_2(x,y)} f(x, y, z)dz$ exists, then*

$$\iiint_D f(x, y, z)dxdydz = \iint_R \left(\int_{f_1(x,y)}^{f_2(x,y)} f(x, y, z)dz \right) dxdy$$

In the above theorem, the region D is bounded below by the surface $z = f_1(x, y)$ and bounded above by the surface $z = f_1(x, y)$, and on the side by the cylinder generated by a line moving parallel to the z-axis along the boundary of R. The projection of D on the xy-plane is the region R.

Example. Let $D = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$, and $f : D \to \mathbb{R}$ is a continuous function, then

$$\iiint_D f(x,y,z)dxdydz = \int_{-1}^{1} \left[\left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} f(x,y,z)dz \right) dy \right] dx.$$

2. Change of Variables in Triple Integrals

The change of variable formula for a double integral can be extended to triple integrals. We will straightaway present the formula.

Formula.

$$\iiint_D f(x,y,z)dxdydz = \iiint_E f[X(u,v,w), Y(u,v,w), Z(u,v,w)]|J(u,v,w)|dudvdw,$$

where the Jacobian determinant $J(u,v,w)$ is defined as follows:

$$J(u,v,w) = \begin{vmatrix} \frac{\partial X}{\partial u} & \frac{\partial Y}{\partial u} & \frac{\partial Z}{\partial u} \\ \frac{\partial X}{\partial v} & \frac{\partial Y}{\partial v} & \frac{\partial Z}{\partial v} \\ \frac{\partial X}{\partial w} & \frac{\partial Y}{\partial w} & \frac{\partial Z}{\partial w} \end{vmatrix}.$$

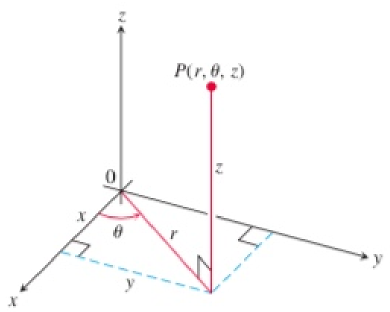
The above formula is valid under the same assumptions we had for the two dimensional case.

3. Triple Integrals in Cylindrical Coordinates

Definition 3. *Cylindrical coordinates represent a point P in space by ordered triples (r, θ, z) in which $r > 0$ and $\theta \in [0, 2\pi)$, and*

(1) r and θ are polar coordinates for the vertical projection of P on the xy-plane,

(2) z is the rectangular vertical coordinate.



The variables x, y and z are changed to r, θ and z by the following three equations

$$x = X(r,\theta) = r\cos\theta, y = Y(r,\theta) = r\sin\theta \text{ and } z = z.$$

The Jacobian is

$$J(u,v,z) = \begin{vmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r.$$

Therefore, the change of variable formula is

$$\iiint_D f(x,y,z)dxdydz = \iiint_E f(r\cos\theta, r\sin\theta, z)rdrd\theta dz.$$

Example. Let us evaluate $\iiint_D (z^2x^2 + z^2y^2)dxdydz$, where D is the region determined by $x^2 + y^2 \le 1, -1 \le z \le 1$.

We can describe D in cylindrical coordinates by $0 \le r \le 1, 0 \le \theta \le 2\pi, -1 \le z \le 1$. Thus,

$$\begin{aligned}\iiint_D (z^2x^2 + z^2y^2)dxdydz &= \int_{-1}^{1}\int_0^{2\pi}\int_0^1 (z^2r^2)rdrd\theta dz \\ &= \int_{-1}^{1}\int_0^{2\pi} z^2\frac{r^4}{4}\Big|_{r=0}^{1} d\theta dz \\ &= \int_{-1}^{1} \frac{2\pi}{4}z^2 dz = \frac{\pi}{3}.\end{aligned}$$

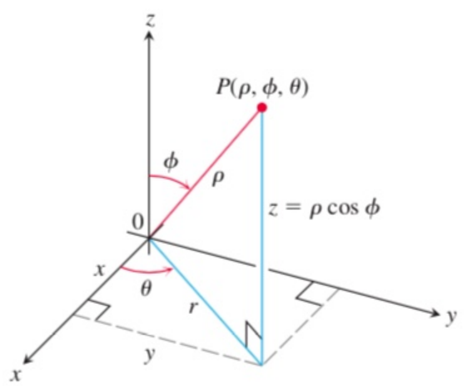
4. Triple Integrals in Spherical Coordinates

Definition 4. *Spherical coordinates represent a point P in space by ordered triples (ρ, ϕ, θ) in which*

(1) ρ is the distance from P to the origin ($\rho \ge 0$),

(2) ϕ is the angle OP makes with the positive z-axis ($0 \le \phi \le \pi$),

(3) θ is the angle from cylindrical coordinates.



Equations Relating Spherical Coordinates to Cartesian Coordinates.

$$r = \rho \sin\phi,\ \ x = r\cos\theta = \rho\sin\phi\cos\theta,$$

$$z = \rho\cos\phi,\ \ y = r\sin\theta = \rho\sin\phi\sin\theta.$$

We keep $\rho > 0, 0 \le \theta < 2\pi$ and $0 \le \phi < \pi$ to get a one-one mapping. The Jacobian determinant is $J(\rho,\phi,\theta) = -\rho^2\sin\phi$. Since $\sin\phi\ \ge 0$, we have $|J(\rho,\phi,\theta)| = \rho^2\sin\phi$ and the change of variable formula is

$$\iiint_D f(x,y,z)dxdydz = \iiint_E f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi d\rho d\theta d\pi.$$

Example. Let $D = \{(x,y,z) : x^2+y^2+z^2 \le 4a^2, z \ge a\}$. Evaluate $\iiint_D \frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}dV$ using spherical coordinates.

If we allow ϕ to vary independently, then ϕ varies from 0 to $\frac{\pi}{3}$ (see Figure 2). If we fix ϕ and allow θ to vary from 0 to 2π, then we obtain a surface of a cone (see Figure 1). Since only a part of the cone is lying in the given region, for a fixed ϕ and θ, ρ varies from $a\sec\phi$ to $2a$ (see Figure 1). Therefore, the integral is

$$\int_0^{\frac{\pi}{3}}\int_0^{2\pi}\int_{a\sec\phi}^{2a}\frac{cos\phi}{\rho^2}|J(\rho,\theta,\phi)|d\rho d\theta d\phi = 2\pi\int_0^{\frac{\pi}{3}}(2a\sin\phi\cos\phi - a\sin\phi)d\phi = \frac{\pi a}{2}.$$