CHANGE OF VARIABLES IN DOUBLE INTEGRALS

We used Fubini's theorem for calculating the double integrals. We have also noticed that Fubini's theorem can be applied if the domain is in a particular form. In this lecture, we will see that in some cases even if the domain is not in that particular form, using change of variables, we can transform the original double integral into another double integral over a new region where we can apply Fubini's theorem. Thus, if f is a real-valued function on a bounded subset D of $\mathbb{R}^2$, and we change the variables x and y to new variables u and v by

$$x = X(u, v) \text{ and } y = Y(u, v) \text{ or collectively } (x, y) = \Phi(u, v),$$

where the map $\Phi(u, v) = (X(u, v), Y(u, v))$ maps a bounded subset E of $\mathbb{R}^2$ onto D, then we would like to see how the double integral $\iint_D f(x, y)dxdy$ in the xy-plane is related to the double integral $\iint_E g(u, v)dudv$ in the uv-plane, where

$$g(u, v) = f(\Phi(u, v)) = f(X(u, v), Y(u, v)).$$

The precise relationship is called the change of variables formula.

Assumptions.

(1) We assume that the mapping Φ from the domain E in the uv-plane to the domain D in the xy-plane is one-one.

(2) The functions X and Y are continuous and have continuous partial derivatives $\frac{\partial X}{\partial u}, \frac{\partial X}{\partial v}, \frac{\partial Y}{\partial u}, \frac{\partial Y}{\partial v}$.

(3) The Jacobian $J(u, v)$ defined below is never zero. The Jacobian is defined as follows:

$$J(u, v) = \begin{vmatrix} \frac{\partial X}{\partial u} & \frac{\partial Y}{\partial u} \\ \frac{\partial X}{\partial v} & \frac{\partial Y}{\partial v} \end{vmatrix}$$

The Change of Variables Formula.

$$\iint_D f(x, y)dxdy = \iint_E f[X(u, v), Y(u, v)]|J(u, v)|dudv.$$

Remark 1. *The change of variables formula is analogous to the principle of Integration by Substitution in one-variable calculus. Let us recall that if* $g : [c, d] \to \mathbb{R}$ *is differentiable,* g' *is integrable on* $[c, d]$ *such that* $g'(t) \neq 0$ *for every* $t \in (c, d)$ *and* $g([c, d]) = [a, b]$*, then* *for any integrable function* $f : [a, b] \to \mathbb{R}$, *the function* $(f \circ g)|g'| : [c, d] \to \mathbb{R}$ *is integrable and*

$$\int_a^b f(x)dx = \int_c^d f(g(t))|g'(t)|dt.$$

If $g'(t) > 0$, *then* g *is strictly increasing and* $g(c) = a, g(d) = b$, *and so the above formula becomes*

$$\int_{g(c)}^{g(d)} f(x)dx = \int_c^d f(g(t))g'(t)dt.$$

Example. Find the area of the region D bounded by the hyperbolas $xy = 1$ and $xy = 2$, and the curves $xy^2 = 3$ and $xy^2 = 4$.

The area of D is $\iint_D dxdy$. Put $u = xy$ and $v = xy^2$, then $x = \frac{u^2}{v}$ and $y = \frac{v}{u}$. The region E is: $1 \leq u \leq 2$ and $3 \leq v \leq 4$. The Jacobian $J(u, v) = \frac{1}{v}$. Therefore, by the change of variable formula,

$$Area(D) = \iint_D dxdy = \iint_E \frac{1}{v}dudv = \int_3^4 \int_1^2 \frac{1}{v}dudv.$$

1. Special case: Polar coordinates

In this case the variables x and y are changed to r and θ by the following two equations: $x = X(r, \theta) = rcos\theta$ and $y = Y(r, \theta) = rsin\theta$. We assume that $r > 0$ and θ lies in $[0, 2\pi)$ so that the mapping involved in the change of variable is one-one. The Jacobian is

$$J(r, \theta) = \begin{vmatrix} \frac{\partial X}{\partial r} & \frac{\partial Y}{\partial r} \\ \frac{\partial X}{\partial \theta} & \frac{\partial Y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r.$$

Therefore, the change of variable formula in this case is:

$$\iint_D f(x, y)dxdy = \iint_E f(r\cos\theta, r\sin\theta)rdrd\theta.$$

Example. Find the volume of the sphere of radius a.

The volume is $V = 2\iint_D \sqrt{a^2 - x^2 - y^2}dxdy$, where $D = \{(x, y) : x^2 + y^2 \leq a^2\}$.

If we use the rectangular coordinates, $V = 2\int\limits_{-a}^{a} \int\limits_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}dydx$ which is complicated to calculate. Let us use the polar coordinates. In polar coordinate, $V = 2\iint_E \sqrt{a^2 - r^2}rdrd\theta$, where $T = [0, a] \times [0, 2\pi]$. By Fubini's theorem,

$$V = 2\int_0^a \int_0^{2\pi} \sqrt{a^2 - r^2}rd\theta dr = 4\pi\int_0^a r\sqrt{a^2 - r^2}dr = \frac{(a^2 - r^2)^{\frac{3}{2}}}{-3}\Bigg|_0^a = \frac{4\pi a^3}{3}.$$