DOUBLE INTEGRALS

In one-variable calculus, we have studied the theory of Riemann integration. In this lecture, we will extend this theory to functions of two variables.

1. Double Integrals on Rectangles

The definition of double integral is similar to the definition of Riemann integral of a single variable function. Consider the rectangle $Q = [a, b] \times [c, d]$, where $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$. Let $f : Q \to \mathbb{R}$ be a bounded function. Let P_1 and P_2 be partitions of $[a, b]$ and $[c, d]$ respectively. Suppose $P_1 = \{x_0, x_1, \ldots, x_m\}$ and $P_2 = \{y_0, y_1, \ldots, y_n\}$. Note that the partition $P = P_1 \times P_2$ decomposes Q into mn sub-rectangles. Define

$$m_{ij} = \inf\{f(x, y) : (x, y) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]\}$$

and

$$M_{ij} = \sup\{f(x, y) : (x, y) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]\},$$

for $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

We define the lower double sum and the upper double sum for the function f with respect to the partition P as follows:

$$L(P, f) = \sum_{i=1}^{m} \sum_{j=1}^{n} m_{ij}(x_i - x_{i-1})(y_j - y_{j-1}) \text{ and } U(P, f) = \sum_{i=1}^{m} \sum_{j=1}^{n} M_{ij}(x_i - x_{i-1})(y_j - y_{j-1}).$$

We now define the **lower double integral** of a bounded function f by

$$L(f) = \sup\{L(P, f) : P \text{ is a partition of } Q\}$$

and the **upper double integral** of a bounded function f by

$$L(f) = \sup\{L(P, f) : P \text{ is a partition of } Q\}.$$

Definition 1. *We say that f is* **integrable** *on Q if $L(f) = U(f)$. The common value is called the* **double integral**, *or simply the* **integral** *of f on Q, and is denoted by*

$$\iint_Q f(x, y)dxdy \text{ or by } \iint_Q f dA.$$

Definition 2. *If f is integrable and nonnegative, then the volume of the solid under the surface given by $z = f(x, y)$ and above the rectangle Q is defined to be the double integral of f over Q, i.e.,*

$$\textit{Volume}(S) = \iint_Q f(x, y)dxdy,$$

where

$$S = \{(x, y, z) \in \mathbb{R}^3 : a \leq x \leq b, c \leq y \leq d \text{ and } 0 \leq z \leq f(x, y)\}.$$

The proof of the following theorem is similar to the single variable case.

Theorem 3. *If a function $f(x, y)$ is continuous on a rectangle $Q = [a, b] \times [c, d]$, then f is integrable on Q.*

Fubini's Theorem on Rectangles

The easiest and the most widely used method to evaluate double integrals is to reduce the problem to a repeated evaluation of Riemann integrals of functions of one variable. The following result shows when and how this can be done.

Theorem 4 (Fubini's Theorem)**.** *Let $f : Q \to \mathbb{R}$ be continuous. Then*

$$\iint_Q f(x, y)dxdy = \int_c^d \left(\int_a^b f(x, y)dx\right) dy = \int_a^b \left(\int_c^d f(x, y)dy\right) dx.$$

2. Double Integrals over Bounded Sets

In this section we extend the theory of double integrals on rectangles developed in the first section to double integrals over an arbitrary bounded subset D of $\mathbb{R}^2$.

Let D be a bounded subset of $\mathbb{R}^2$ and let $f : D \to \mathbb{R}$ be a bounded function. Consider a rectangle Q such that $D \subseteq Q$. Define a function $f^* : Q \to Q$ by

$$f^*(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \in D, \\ 0 & \text{otherwise.} \end{cases}$$

We say that f is integrable over D if f^* is integrable on Q, and in this case, the double integral of f (over D) is defined to be the double integral of f^* (on Q), that is,

$$\iint_D f(x, y)dxdy = \iint_Q f^*(x, y)dxdy.$$

Remark 5. *We can show that the integrability of f over D and the value of its double integral are independent of the choice of a rectangle Q containing D and the corresponding extension f^* of f to Q.*

Definition 6. *If D is a bounded subset of $\mathbb{R}^2$ and $f : D \to \mathbb{R}$ is integrable and nonnegative,, then the volume of the solid under the surface given by $z = f(x, y)$ and above the region D is defined to be the double integral of f over D, i.e.,*

$$Volume(S) = \iint_D f(x, y)dxdy,$$

where

$$S = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in D \text{ and } 0 \leq z \leq f(x, y)\}.$$

Fubini's Theorem over Elementary Regions

We have seen a useful method of evaluating a double integral on a rectangle by converting it to an iterated integral. The relevant result of Fubini, when generalized to some special regions of $\mathbb{R}^2$ which are called elementary regions, yields the most convenient way to calculate double integrals.

Let $f_1, f_2 : [a, b] \to \mathbb{R}$ such that f_1 and f_2 are Riemann integrable, $f_1 \leq f_2$, and

$$D_1 = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b \text{ and } f_1(x) \leq y \leq f_2(x)\},$$

and let $g_1, g_2 : [c, d] \to \mathbb{R}$ such that g_1 and g_2 are Riemann integrable, $g_1 \leq g_2$, and

$$D_2 = \{(x, y) \in \mathbb{R}^2 : c \leq y \leq d \text{ and } g_1(x) \leq x \leq g_2(x)\},$$

then D_1 and D_2 are called elementary regions.

Theorem 7 (Fubini's Theorem (Stronger form))**.** *(1) Let f be an integrable function the elementary region D_1. If for each fixed $x \in [a, b]$, the Riemann integral $\int\limits_{f_1(x)}^{f_2(x)} f(x, y)dy$ exists, then*

$$\iint_D f(x, y)dxdy = \int_a^b \left(\int_{f_1(x)}^{f_2(x)} f(x, y)dy \right) dx.$$

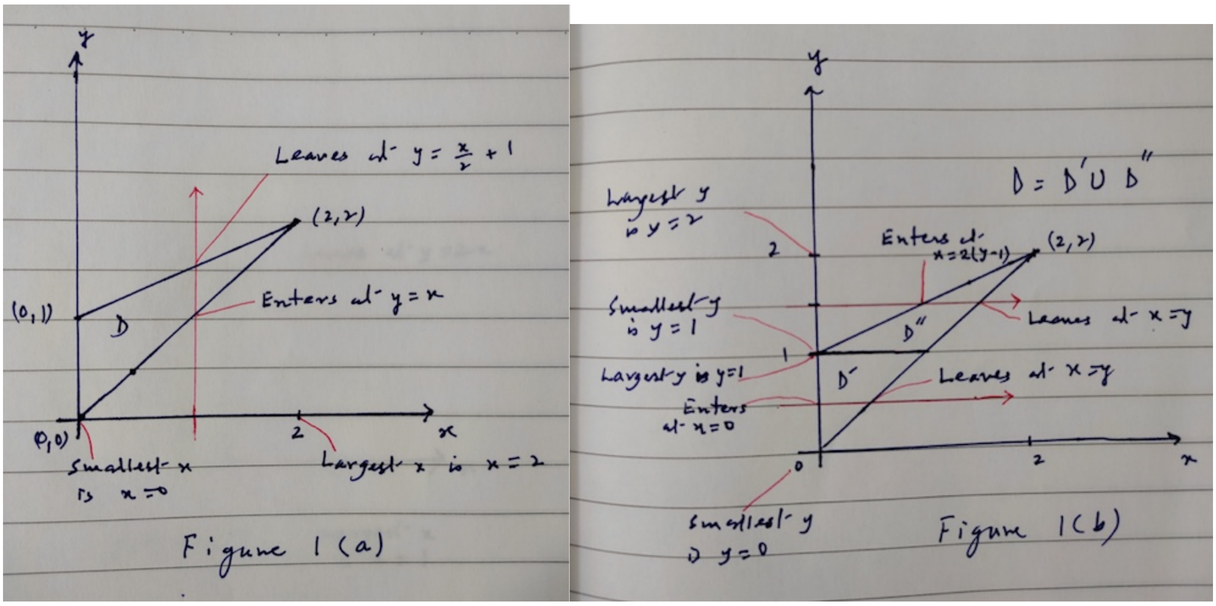
(2) Let f be an integrable function the elementary region D_2. If for each fixed $y \in [c, d]$, the Riemann integral $\int\limits_{g_1(x)}^{g_2(x)} f(x, y)dx$ exists, then

$$\iint_D f(x, y)dxdy = \int_c^d \left(\int_{g_1(x)}^{g_2(x)} f(x, y)dx \right) dy.$$

Example. Evaluate the integral $\iint_D (x+y)^2 dxdy$, where D is the region bounded by the lines joining the points $(0,0),(0,1)$ and $(2,2)$.

The domain D (see Figure 1(a)) is like the elementary region D_1 above, i.e., $D = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 2 \text{ and } x \le y \le \frac{x}{2}+1\}$. Therefore, by second part of Theorem 7,

$$\iint_D (x+y)^2 dxdy = \int_0^2 \left(\int_x^{\frac{x}{2}+1} (x+y)^2 dy \right) dx.$$



We evaluated the integral in the order $dydx$. What if we change the order of integration to $dxdy$. Then we have to apply first part of Theorem 7. The region of integration becomes $D = D' \cup D'' = \{(x,y) \in \mathbb{R}^2 : 0 \le y \le 1 \text{ and } 0 \le x \le y\} \cup \{(x,y) \in \mathbb{R}^2 : 1 \le y \le 2 \text{ and } 2(y-1) \le x \le y\}$ (see Figure 1(b)). Thus,

$$\iint_D (x+y)^2 dxdy = \int_0^1 \left(\int_0^y (x+y)^2 dx \right) dy + \int_1^2 \left(\int_{2(y-1)}^y (x+y)^2 dx \right) dy.$$

Verify that the value of the integral in both cases are equal.

Example. Evaluate $\int\limits_0^2 \left(\int\limits_{y/2}^1 e^{x^2} dx \right) dy$.

The region D of integration (see Figure 2) is given by $D = \{(x,y) \in \mathbb{R}^2 : 0 \le y \le 2, \frac{y}{2} \le x \le 1\}$. Here, we are given two consecutive single integrals. First we have to integrate w.r.t. x and then w.r.t. y. However, the integral $\int\limits_{y/2}^1 e^{x^2} dx$ cannot be evaluated in terms of simple known functions. So, we will use Fubini's theorem and change the order of integration, i.e., from $dxdy$ to $dydx$. Note that when we change the order of integration the limits will change. Then the region of integration becomes $D = \{(x,y) \in \mathbb{R}^2 : 0 \le$

$x \leq 1, 0 \leq x \leq 2x\}$ (see Figure 2). Thus, by first part of Theorem 7,

$$\iint_D f(x,y)dydx = \int_0^1 \left(\int_0^{2x} e^{x^2} dy \right) dx = e - 1.$$