LOCAL EXTREMA AND SADDLE POINTS

In one-variable calculus, we applied the notion of differentiation to study local and global extrema of real-valued functions of one variable. In this lecture we shall see similar applications of the notion of differentiation to functions of two (or more) variables.

Absolute and Critical Points

We have seen that a continuous real-valued function defined on a closed and bounded subset of $\mathbb{R}^2$ is bounded and attains its bounds. That is, if $D \subset \mathbb{R}^2$ is closed and bounded, and $f : D \to \mathbb{R}$ is continuous, then the absolute minimum and the absolute maximum of f on D, namely,

$$m = \inf\{f(x, y) : (x, y) \in D\} \text{ and } M = \sup\{f(x, y) : (x, y) \in D\}$$

exist. Moreover, there are points (x_1, y_1), $(x_2, y_2) \in D$ such that $m = f(x_1, y_1)$ and $M = f(x_2, y_2)$. Thus,

$$m = \min\{f(x, y) : (x, y) \in D\} \text{ and } M = \max\{f(x, y) : (x, y) \in D\}$$

The definition of local extrema is similar to case of real-valued functions. Recall that for $(x, y) \in \mathbb{R}^2$ and $r > 0$, $B((x, y), r)$ is an open disc (ball) of radius r centered at (x, y). Moreover,

Definition 1. *Let $D \subseteq \mathbb{R}^2$ and let (x_0, y_0) be an interior point of D. We say that $f : D \to \mathbb{R}$ has local maximum (minimum) at (x_0, y_0) if there exists $r > 0$ such that $B((x_0, y_0), r) \subseteq D$ and $f(x, y) \le f(x_0, y_0)$ $(f(x, y) \ge f(x_0, y_0))$ for all $(x, y) \in B((x_0, y_0), r)$.*

Definition 2. *Given $D \subseteq \mathbb{R}^2$ and $f : D \to \mathbb{R}$, an interior point (x_0, y_0) of D is called a* **critical point** *if either $\nabla f(x_0, y_0)$ does not exist, or $\nabla f(x_0, y_0)$ exists and $\nabla f(x_0, y_0) = 0$. Recall that $\nabla f(x_0, y_0) = \left(\frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0)\right)$.*

Theorem 3 (Necessary condition for local extremum)**.** *Let $D \subseteq \mathbb{R}^2$ and let (x_0, y_0) be an interior point of D. Suppose $f : D \to \mathbb{R}$ has a local extremum at (x_0, y_0). If $\nabla f(x_0, y_0)$ exists, then $\nabla f(x_0, y_0) = 0$.*

Example. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = xy$. Then $\nabla f(x, y) = (y, x)$ for all $(x, y) \in \mathbb{R}^2$. The only point where f can possibly have a local extremum is $(0, 0)$. We observe that $f(0, 0) = 0$ and for any $r > 0$, there are (x_1, y_1), $(x_2, y_2) \in B((0, 0), r)$ such that $f(x_1, y_1) < 0$ and $(x_2, y_2) > 0$ (Can you find such points?). It follows that f has neither a local maximum nor a local minimum at $(0,0)$. Such a point is called a saddle point.

Definition 4. *Given $D \subseteq \mathbb{R}^2$ and $f : D \to \mathbb{R}$, an interior point (x_0, y_0) of D is called a* **saddle point** *if $\nabla f(x_0, y_0) = 0$ but f does not have a local extremum at (x_0, y_0).*

Example. Let $D = [-2,2] \times [-2,2]$ and $f : D \to \mathbb{R}$ be given by $f(x,y) = 4xy - 2x^2 - y^4$. Let us find the absolute maxima and absolute minima of f in D. Since f is continuous and D is closed and bounded, the absolute extrema of f exist and are attained by$f$. To find out these, we consider the partial derivatives at interior points of D. In particular,

$$\nabla f(x,y) = (4y - 4x, 4x - 4y^3) = (0,0) \Rightarrow (x,y) = (0,0),\ (1,1),\ \text{or } (-1,-1).$$

Also, $(x,y) \in \partial D$ if and only if $x = \pm 2$ or $y = \pm 2$. So we have to consider the functions $f(2,y), f(-2,y), f(x,-2)$ and $f(x,2)$ from $[-2,2]$ to $\mathbb{R}$. Since $f(-x,-y) = f(x,y)$ it is sufficient to consider the functions $f(2,y)$ and $f(x,2)$ only.

For $f(2,y) = 8y - 8 - y^4, y \in [-2,2]$, has absolute maximum at $y = \sqrt[3]{2}$ and absolute minimum at $y = -2$. Similarly, $f(x,2) = 8x - 2x^2 - 16$, $x \in [-2,2]$ has absolute maximum at $x = 2$ and absolute minimum at $x = -2$. It follows that the absolute maximum of f on D is 1 which is attained at $(1,1)$ as well as at $(-1,-1)$. The absolute minimum of f on D is -40 which is attained at $(2,-2)$ as well as at $(-2,2)$.

Second Derivative Test or Discriminant Test for Local Extrema

Let $D \subseteq \mathbb{R}^2$ and let (x_0, y_0) be an interior point of D. Suppose $f : D \to \mathbb{R}$ is such that the first-order and second-order partial derivatives of f exist and are continuous in $B((x_0,y_0), r)$ for some $r > 0$ with $B((x_0,y_0), r) \subseteq D$. We define the **discriminant** of f at (x_0, y_0) to be

$$\Delta f(x_0, y_0) = f_{xx}(x_0,y_0) f_{yy}(x_0,y_0) - [f_{xy}(x_0,y_0)]^2$$

With these assumptions we have the following result.

Theorem 5 (Discriminant Test)**.** *Let $\nabla f(x_0, y_0) = (0,0)$.*

(1) If $\Delta f(x_0,y_0) > 0$ and $f_{xx}(x_0,y_0) < 0$, then f has a local maximum at (x_0,y_0).
(2) If $\Delta f(x_0,y_0) > 0$ and $f_{xx}(x_0,y_0) > 0$, then f has a local minimum at (x_0,y_0).
(3) If $\Delta f(x_0,y_0) < 0$, then f has a saddle point at (x_0,y_0).

Examples.

(1) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = 4xy - x^4 - y^4$. Then f has continuous partial derivatives of all orders. Also,

$$\nabla f(x, y) = (4(y - x^3), 4(x - y^3)) = (0, 0) \iff (x, y) = (0, 0),\ (1, 1), \text{or } (-1, -1).$$

Further, $f_{xx} = -12x^2$, $f_{xy} = 4$ and $f_{yy} = -12y^2$. The discriminant is given by $\Delta f(x, y) = 16(9x^2y^2 - 1)$. Hence, $\Delta f(0, 0) = -16 < 0$ and $\Delta f(1, 1) = \Delta f(-1, -1) = 128 > 0$. Moreover, $f_{xx}(1, 1) = f_{xx}(-1, -1) = -12 < 0$. By the Discriminant Test, f has a saddle point at $(0, 0)$ and a local maximum at $(1, 1)$ and $(-1, -1)$.

(2) Consider the functions $f(x, y) = -(x^4 + y^4)$. Then $\nabla f(x, y) = (-4x^3, -4y^3) = (0, 0) \iff (x, y) = (0, 0)$ but $\Delta f(0, 0) = 0$. So, the Discriminant Test is not applicable to f at $(0, 0)$. We can see directly that has a local maximum at $(0, 0)$. Similarly if $g(x, y) = x^4 + y^4$, $\nabla g(x, y) = (4x^3, 4y^3) = (0, 0) \iff (x, y) = (0, 0)$ but $\Delta g(0, 0) = 0$. So, the Discriminant Test is not applicable. Likewise, we can see directly that has a local minimum at $(0, 0)$.