HIGHER ORDER PARTIAL DERIVATIVES

Consider a function $f : \mathbb{R}^2 \to \mathbb{R}$. If $f_x(x_0, y_0)$ exists at every $(x_0, y_0) \in \mathbb{R}^2$, we obtain a function $f_x : \mathbb{R}^2 \to \mathbb{R}$. So, we can consider the partial derivative of f_x w.r.t. x and y. These partial derivatives, if exist at (x_0, y_0), are denoted by $f_{xx}(x_0, y_0)$ and $f_{xy}(x_0, y_0)$ respectively. In case f_y is defined on $\mathbb{R}^2$, we can similarly define the partial derivative of f_y w.r.t. x and y, and if they exist at (x_0, y_0), they are denoted by $f_{yx}(x_0, y_0)$ and $f_{yy}(x_0, y_0)$ respectively. These partial derivatives are sometimes denoted by

$$\frac{\partial^2 f}{\partial x^2}(x_0, y_0), \ \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0), \ \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0), \ \text{ and } \frac{\partial^2 f}{\partial y^2}(x_0, y_0)$$

instead of $f_{xx}(x_0, y_0)$, $f_{xy}(x_0, y_0)$, $f_{yx}(x_0, y_0)$ and $f_{yy}(x_0, y_0)$, respectively. Collectively, these are referred to as the **second-order partial derivatives** or simply the second partials of f at (x_0, y_0). Among these, the middle two, namely, $f_{xy}(x_0, y_0)$ and $f_{yx}(x_0, y_0)$ are called the **mixed (second-order) partial derivatives of** f, or simply the mixed partials of f.

The order in which x and y appear in mixed partial derivatives can sometimes be a matter of confusion. The order may be easier to remember if one notes that

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} \text{ and } f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}.$$

Example. Let

$$f(x, y) = \begin{cases} xy\frac{x^2 - y^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

We note that

$$f_{xy}(0, 0) = \lim_{k \to 0} \frac{f_x(0, k) - f_x(0, 0)}{k} \text{ and } f_{yx}(0, 0) = \lim_{h \to 0} \frac{f_y(h, 0) - f_y(0, 0)}{h}.$$

Now,

$$f_x(0, k) = \lim_{h \to 0} \frac{f(h, k) - f(0, k)}{h} = \lim_{h \to 0} \frac{k(h^2 - k^2)}{h^2 + k^2} = -k,$$

and

$$f_y(h, 0) = \lim_{k \to 0} \frac{f(h, k) - f(h, 0)}{k} = \lim_{k \to 0} \frac{h(h^2 - k^2)}{h^2 + k^2} = h.$$

Therefore,

$$f_{xy}(0, 0) = \lim_{k \to 0} \frac{-k - 0}{k} = -1 \text{ and } f_{yx}(0, 0) = \lim_{h \to 0} \frac{h - 0}{h} = 1.$$

So, in general f_{xy} may not be equal to f_{yx}. We get equality of mixed second-order partial derivatives if one of them is continuous. In fact,

Theorem 1 (Mixed Partials Theorem)**.** *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that both f_x and f_y exist in a neighborhood B of $(x_0, y_0) \in \mathbb{R}^2$. If f_{xy} or f_{yx} exists on B and is continuous at (x_0, y_0), then both f_{xy} oand f_{yx} exist and*

$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0).$$

Remark 2. *In the above example, we can verify that f_x, f_y, f_{xy} and f_{yx} exist on $\mathbb{R}^2 \setminus \{(0,0)\}$. Thus, from the above Theorem, it follows that neither f_{xy} nor f_{yx} can be continuous at $(0,0)$.*