LIMITS AND CONTINUITY OF FUNCTIONS OF SEVERAL VARIABLES

In this lecture we will discuss notion of limit and continuity of functions $f : \mathbb{R}^2 \to \mathbb{R}$. The same idea can be generalized to higher dimensions.

Definition 1 (Continuity). *Let $D \subseteq \mathbb{R}^2$, $(x_0, y_0) \in D$, and let $f : D \to \mathbb{R}$ be any function. f is continuous at (x_0, y_0) if for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$|f(x, y) - f(x_0, y_0)| < \varepsilon \text{ whenever } ||(x, y) - (x_0, y_0)|| < \delta.$$

The following theorem gives the sequential criterion for Continuity.

Theorem 2. *The function f is continuous at (x_0, y_0) if and only if for every sequence (x_n, y_n) in D such that $(x_n, y_n) \to (x_0, y_0)$, we have $f(x_n, y_n) \to f(x_0, y_0)$.*

Definition 3 (Limit). *Let $D \subseteq \mathbb{R}^2$, $(x_0, y_0) \in \mathbb{R}^2$, and let $f : D \to \mathbb{R}$ be any function. Assume that there is $r > 0$ such that $B((x_0, y_0), r) \setminus \{(x_0, y_0)\} \subseteq D$. Then the limit of f as $(x, y) \to (x_0, y_0)$ is ℓ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$|f(x, y) - \ell| < \varepsilon \text{ whenever } 0 < ||(x, y) - (x_0, y_0)|| < \delta.$$

We write this as $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = \ell$ or $f(x, y) \to \ell$ as $(x, y) \to (x_0, y_0)$.

The sequential criterion for limit is given below.

Theorem 4. *$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = \ell$ if and only if for every sequence (x_n, y_n) in $D \setminus \{(x_0, y_0)\}$ such that $(x_n, y_n) \to (x_0, y_0)$, we have $f(x_n, y_n) \to \ell$.*

The concepts of continuity and limit are related in a similar way as in the case of functions of one variable.

Proposition 5. *Let $D \subseteq \mathbb{R}^2$, (x_0, y_0) be an interior point of D, and let $f : D \to \mathbb{R}$ be any function. Then f is continuous at (x_0, y_0) if and only if $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$.*

Examples.

(1) If $f : \mathbb{R}^2 \to \mathbb{R}$ is a constant function, then f is continuous.

(2) If $f : \mathbb{R}^2 \to \mathbb{R}$ is given by $f(x, y) = \sqrt{x^2 + y^2}$. Then f is continuous on $\mathbb{R}^2$. Let $(x_n, y_n) \to (x, y)$. Then $x_n \to x$ and $y_n \to y$. This implies that $f(x_n, y_n) = \sqrt{x_n^2 + y_n^2} \to \sqrt{x^2 + y^2}$.

(3) The coordinate functions $p_1, p_2 : \mathbb{R}^2 \to \mathbb{R}$ defined $p_1(x, y) = x$ and $p_2(x, y) = y$ are continuous.

(4) Consider $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(0,0) = 0$ and $f(x,y) = \sin xy$ for $(x,y) \neq (0,0)$. Then $\lim_{(x,y)\to(0,0)} f(x,y) = 0$. Indeed, if $((x_n, y_n) \in \mathbb{R}^2 \setminus \{(0,0)\}$ is a sequence such that $(x_n, y_n) \to (0,0)$, then $x_n y_n \to 0$, and hence $\sin(x_n y_n) \to \sin 0 = 0$. Since $f(0,0) = 0$, f is continuous at $(0,0)$.

(5) Consider $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} x + y & \text{if } x \neq y, \\ 1 & \text{if } x = y. \end{cases}$$

Then $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist. Consider $(x_n, y_n) = (1/n, 1/n)$ and $(z_n, w_n) = (-1/n, 1/n)$. Both these sequences converge to $(0,0)$ but $f(x_n, y_n) \to 1$ and $f(z_n, w_n) \to 0$.

(6) Consider $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ defined by $f(x,y) = \frac{xy}{x^2+y^2}$. Then $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist. Consider $(x_n, y_n) = (1/n, 1/n)$ and $(z_n, w_n) = (-1/n, 2/n)$. Both these sequences converge to $(0,0)$ but $f(x_n, y_n) \to 1/2$ and $f(z_n, w_n) \to 2/5$.

(7) Let $f(x,y) = \frac{x^2 y}{x^4+y^2}$ when $(x,y) \neq (0,0)$ and $f(0,0) = 0$. Then f is not continuous at $(0,0)$ because $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist. Observe that $f(1/n, 1/n) \to 0$ and $f(1/n, 1/n^2) \to 1/2$.

(8) Let $f(x,y) = \frac{x^4 - y^2}{x^4+y^2}$ when $(x,y) \neq (0,0)$ and $f(0,0) = 0$. Show that f is not continuous at $(0,0)$.