THE REAL NUMBER SYSTEM

In this lecture we shall study some properties of real numbers. We are already familiar with the set of natural numbers (or positive integers) $\mathbb{N} = \{1, 2, 3, \ldots\}$[1]. The set of integers $\mathbb{Z}$[2] consists of positive integers, 0, and negative integers, i.e., $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. Quotients of integers are called rational numbers denoted by $\mathbb{Q}$, that is, $\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0\}$.

We can represent integers by points on a straight line (called the number line) by fixing the number 0 and a unit distance. By subdividing the segment between 0 and 1, we can represent the rational numbers $\frac{1}{n}$, where $n \in \mathbb{N}$. Thus, we can use this to represent any rational number by a unique point on the number line. Though rational numbers seems to fill the number line, there are some gaps. For example, the number $\sqrt{2}$ is not a rational number (can be easily proved). Similarly, numbers like π and e are also irrationals (not so easy to prove!). Such numbers are called irrational numbers. The rational numbers and the irrational numbers together constitute the set $\mathbb{R}$.

But what exactly are irrational numbers? Or for that matter real numbers? We haven't actually given the precise (mathematical) definition of real numbers. To define the real numbers, one begins with the set $\mathbb{Q}$ and construct the set $\mathbb{R}$. There are two standard approaches for the construction of $\mathbb{R}$ from $\mathbb{Q}$, one due to Dedekind (through Dedekind cuts) [3], and the other due to Cantor (through Cauchy sequences) [1]. Both Cantor and Dedekind published their construction in 1872. We will not study these constructions in this course because they are little complicated, rather we will study some other useful and important properties of real numbers.

We already know that the set of real numbers $\mathbb{R}$ is a field under usual addition and multiplication. Moreover, under usual addition and scalar multiplication, $\mathbb{R}(\mathbb{R})$ is a vector space. We also know that the set of rational numbers $\mathbb{Q}$ is a subfield of $\mathbb{R}$ and it is also a subspace of $\mathbb{R}(\mathbb{Q})$.

Order Properties

We refer to the vector space and field properties of real numbers as algebraic properties. The set $\mathbb{R}$ contains a subset $\mathbb{R}^+$, called the set of all positive real numbers, satisfying the following properties:

[1] In some texts the natural numbers start at 0 instead of 1. This is just a matter of notational convention. The first evidence we have of zero is from the Sumerian culture in Mesopotamia, some 5000 years ago. For more details see [2].

[2] The letter $\mathbb{Z}$ stands for the German word *Zahlen* for numbers.

(1) Given any $x \in \mathbb{R}$, exactly one of the following statements is true:

$$x \in \mathbb{R}^+;\ x = 0;\ -x \in \mathbb{R}^+.$$

(2) If $x, y \in \mathbb{R}^+$, then $x + y \in \mathbb{R}^+$ and $xy \in \mathbb{R}^+$.

We define an order relation on $\mathbb{R}$ as follows:

For $x, y \in \mathbb{R}$, we define x to be less than y, we write $x < y$, if $y - x \in \mathbb{R}^+$. We also write $y > x$ and say that y is greater than x. It follows that $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$. Using the algebraic properties of $\mathbb{R}$ and the properties (1) and (2) above we can easily prove the following:

(i) Given any $x, y \in \mathbb{R}$, one of the following is true.

$$x < y;\ x = y,\ y < x.$$

This is called the **Law of Trichotomy**.

(ii) If $x, y, z \in \mathbb{R}$ such that $x < y$ and $y < z$, then $x < z$.

(iii) If $x, y, z \in \mathbb{R}$ such that $x < y$, then $x + z < y + z$. Moreover, if $z > 0$, then $xz < yz$, whereas if if $z < 0$, then $xz > yz$.

The notation $x \leq y$ means that either $x < y$ or $x = y$. Likewise, $x \geq y$ means $x > y$ or $x = y$.

Completeness Property

We begin with the following definitions.

Definition 1 (Bounded above and below)**.** *Let S be a nonempty subset of $\mathbb{R}$.*

- *We say that S is* **bounded above** *if there exists $\alpha \in \mathbb{R}$ such that $x \leq \alpha$ for all $x \in S$. Any such α is called an* **upper bound** *of S. Express this definition in terms of quantifiers.*
- *We say that S is* **bounded below** *if there exists $\beta \in \mathbb{R}$ such that $x \geq \beta$ for all $x \in S$. Any such β is called a* **lower bound** *of S.*
- *The set S is said to be* **bounded** *if it is bounded above as well as bounded below; otherwise, S is said to be unbounded.*

Examples.

(1) $\mathbb{N}$ is bounded below, and any real number $\beta \leq 1$ is a lower bound of $\mathbb{N}$.

(2) The set $S = \{\frac{1}{n} : n \in \mathbb{N}\}$ is bounded. Indeed, any real number $\alpha \geq 1$ is an upper bound of S, whereas any real number $\beta \leq 0$ is a lower bound of S.

(3) The set $S = \{x \in \mathbb{Q} : x^2 < 2\}$ is bounded. Here, 2 is an upper bound, while -2 is a lower bound.

Exercise 2. *When do you say a real number is not a lower bound of S? When do you say S is not bounded below? Express these in terms of quantifiers.*

Definition 3 (Supremum and Infimum)**.** *Let S be a nonempty subset of $\mathbb{R}$.*

- *An element $M \in \mathbb{R}$ is a* **supremum** *or a* **least upper bound** *of S if*
  - *(i) M is an upper bound of S, and*
  - *(ii) If α is an upper bound of S, then $M \leq \alpha$.*

  The symbol LUB is a shorthand notation for least upper bound. We denote the supremum of S by $\sup S$ or $\operatorname{lub} S$.
- *An element $m \in \mathbb{R}$ is a* **infimum** *or a* **greatest lower bound** *of S if*
  - *(i) m is a lower bound of S, and*
  - *(ii) If β is a lower bound of S, then $m \geq \beta$.*

  The symbol GLB is a shorthand notation for greatest lower bound. We denote the infimum of S by $\inf S$ or $\operatorname{glb} S$.
- *$\sup S$ and $\inf S$ may belong to the set S. If $\sup S \in S$, it is called the maximum of S, and denoted by $\max S$; likewise, if $\inf S \in S$, it is called the minimum of S, and denoted by $\min S$.*

Exercise 4. *Prove that if S has a supremum, it must be unique. The same assertion holds for infimum as well.*

Problem 5. *Find the supremum and infimum of the set $S = \{x + x^{-1} : x > 0\}$.*

Solution. First observe that if $x > 0$, $x + \frac{1}{x} > x$. Hence, S is not bounded above.

On the other hand, if $x > 0$, $(x-1)^2 \geq 0 \implies x^2 - 2x + 1 \geq 0$. Dividing by x we get, $x - 2 + \frac{1}{x} \geq 0$ or $2 \leq x + \frac{1}{x}$. Thus, 2 is a lower bound of S. Further, note that $2 = 1 + \frac{1}{1} \in S$. This implies that $\inf S = 2$.

The following characterization of the LUB will be very useful.

Proposition 6. *Let S be a nonempty subset of $\mathbb{R}$. Then $M = \sup S$ if and only if*

(1) M is an upper bound of S,

(2) If $\beta < M$, then β is not an upper bound of S.

Proof. The direct part is clear. For the converse part, let α be an upper bound of S. We claim that $M \leq \alpha$. If not, then $M > \alpha$. So, by the hypothesis (2), α is not an upper bound, contradicting our assumption. Therefore, $M \leq \alpha$, and $M = \sup S$. □

Exercise 7. *State and prove an analogous to the above Proposition for GLB.*

It is time to state the most important property of $\mathbb{R}$ which is called the **completeness property** or the **LUB property**. It says that

"Every nonempty subset of $\mathbb{R}$ that is bounded above has a LUB".

Theorem 8. *Let S be a nonempty subset of $\mathbb{R}$ that is bounded below. Then S has an infimum.*

Proof. Let $A = \{\alpha \in \mathbb{R} : \alpha \leq x, \ \forall \ x \in S\}$. Since S is bounded below, $A \neq \emptyset$. Let $x \in S$. By definition of the set A, $\beta \leq x$ for all $\beta \in A$. Then, x is an upper bound of A. This implies that $\sup A \leq x$ ($\sup A$ exists because A is bounded above). Thus, $\sup A$ is a lower bound of S. We claim that $\sup A = \inf S$. Indeed, if α is any lower bound of S, then $\alpha \in A$. Therefore, $\alpha \leq \sup A$, which completes the proof. □

The following theorem is an important consequence of the completeness property of $\mathbb{R}$.

Theorem 9 (Archimedean property (AP))**.** *Given $x, y \in \mathbb{R}$ with $x > 0$, there exists $n \in \mathbb{N}$ such that $nx > y$.*

Proof. Proof by contradiction. If false, then $nx \leq y$ for every $n \in \mathbb{N}$. Thus, y is an upper bound of the set $S = \{nx : n \in \mathbb{N}\}$. Let $M = \sup S$. Now for all n, $(n+1)x \leq M$ or $nx \leq M - x < M$. Hence, $M - x$ is an upper bound of S, a contradiction. □

Corollary 10. *$\mathbb{N}$ is not bounded above in $\mathbb{R}$, that is, for any $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > x$.*

Example 11. *Find the supremum and infimum of the set $S = \left\{\frac{m}{|m|+n} : m \in \mathbb{Z}, \ n \in \mathbb{N}\right\}$.*

Solution. Observe that $-1 < \frac{m}{|m|+n} < 1$, and for $m \in \mathbb{Z}$, $\frac{m}{|m|+1} \in S$. For $m > 0$, $\frac{m}{m+1}$ approaches 1 (whatever that means! We will see the precise definition in the next lecture), and $m < 0$, $\frac{m}{-m+1}$ approaches -1. Hence, we guess that $\sup S = 1$ and $\inf S = -1$. Let's prove this.

If $\beta < 1$, $1 - \beta > 0$. By AP (take $x = 1 - \beta, \ y = \beta$), $\exists \ n \in \mathbb{N}$ such that $n(1-\beta) > \beta$ $\Rightarrow \beta < \frac{n}{n+1} \in S$. Thus, $\sup S = 1$.

If $\alpha > -1$, $1+\alpha > 0$. By AP (take $x = 1+\alpha, \ y = -\alpha$), $\exists \ n \in \mathbb{N}$ such that $n(1+\alpha) > -\alpha$ $\Rightarrow \frac{-n}{n+1} < \alpha$. Hence, $\inf S = -1$.

References

[1] E. Hewitt and K. Stromberg, *Real and Abstract Analysis*, Springer-Verlag, New York, 1965.

[2] R. Kaplan, *The Nothing That Is: A Natural History of Zero*, Oxford University Press, New York, 2000.

[3] W. Rudin, *Principles of Mathematical Analysis*, third ed., McGraw-Hill, New York-Auckland-Düsseldorf, 1976.