THE EUCLIDEAN SPACES

Algebraic and Metric Structure of $\mathbb{R}^n$

In our study of this course so far, we have focussed on real-valued functions of one variable, i.e., functions $f : D \to \mathbb{R}$, where D is a subset of $\mathbb{R}$. We have seen that the set of real numbers $\mathbb{R}$ is more than just a collection of points. It has various algebraic and order structures that play an important role in the development of our subject. The same is true for the n-dimensional Euclidean space $\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) : x_1, x_2, \dots, x_n \in \mathbb{R}\}$ which we will study in this and the coming lectures. We will also study the properties of functions of several variables, i.e., functions $f : \mathbb{R}^n \to \mathbb{R}^m$.

In the course of Linear Algebra, we have seen that $\mathbb{R}^n$ is vector space over $\mathbb{R}$ under the operations

$$(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n) \text{ and } \alpha(x_1, \dots, x_n) = (\alpha x_1, \dots, \alpha x_n),$$

for all $(x_1, \dots, x_n), (y_1, \dots, y_n) \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$.

We have also seen that the vector space $\mathbb{R}^n(\mathbb{R})$ is an inner product space with the inner product (dot product)

$$\langle (x_1, \dots, x_n), (y_1, \dots, y_n) \rangle = x_1 y_1 + \cdots x_n y_n.$$

The norm of a vector $x = (x_1, \dots, x_n)$ is defined as $||x|| = \sqrt{x_1^2 + \cdots + x_n^2}$. The distance between $x, y \in \mathbb{R}^n$ is defined as $d(x, y) = ||x - y||$.

We recall the following important inequality.

Theorem 1 (Cauchy-Schwartz Inequality)**.** *Let $x, y \in \mathbb{R}^n$. Then $|\langle x, y \rangle| \leq ||x||||y||$.*

It follows form the Cauchy-Schwartz Inequality that if $x, y \in \mathbb{R}^n$ are nonzero vectors, then $-1 \leq \frac{|\langle x,y \rangle|}{||x||||y||} \leq 1$. Any number in $[-1, 1]$ is the cosine of a unique angle between 0 and π. The following definition is anticipated.

Definition 2. *The angle θ between x and y is defined to be*

$$\theta = \cos^{-1} \left(\frac{|\langle x, y \rangle|}{||x||||y||} \right).$$

Elementary Topology of $\mathbb{R}^n$

We start with generalizing the notion of intervals.

Definition 3. *Let $x \in \mathbb{R}^n$ and let $r > 0$. The set*

$$B(x,r) = \{y \in \mathbb{R}^n : ||y - x|| < r\}$$

is called the **open ball of radius r centered at x.**

This set consist of those points whose distance from x is less than r. How $B(x,r)$ looks like in $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$?

Definition 4. *A sequence in $\mathbb{R}^n$ is a function $f : \mathbb{N} \to \mathbb{R}^n$.*

As was the case for sequences in $\mathbb{R}$, we let $x_k = f(k)$ and we will use the notation (x_k) to denote sequences in $\mathbb{R}^n$. Since each x_k is a vector in $\mathbb{R}^n$, we can write x_k using double subscript notation $x_k = (x_{k1}, \ldots, x_{kn})$.

Examples. Some sequences (x_k) in $\mathbb{R}^2$ and $\mathbb{R}^3$.

a) $x_k = (e^k, k^2)$ b) $x_k = (\frac{1}{k}, \frac{1}{k!})$ c) $x_k = (\cos\frac{\pi}{k}, \sin\frac{\pi}{k})$

d) $x_k = (\frac{\log k}{k}, \frac{k^2}{e^k}, (-1)^k)$ e) $x_k = (\sqrt{k+1} - \sqrt{k}, \frac{1}{e}, k\pi)$ f) $x_k = (1, -1, \sqrt{2})$

Definition 5. *Let (x_k) be a sequence in $\mathbb{R}^n$,*

(1) (x_k) is called bounded if there exists $M > 0$ such $||x_k|| \leq M$ for all $k \in \mathbb{N}$.

(2) We say that (x_k) converge to a point $x \in \mathbb{R}^n$ if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $||x_k - x|| < \varepsilon$ for all $k \geq N$. In this case, we write $\lim_{k\to\infty} x_k = x$ or $x_k \to x$ as $k \to \infty$ or simply $x_k \to x$.

A sequence that is not convergent is said to be divergent.

Definition 6. *Let $E \subseteq \mathbb{R}^n$.*

(1) A point $x \in E$ is called an **interior point** *of E if there exists $r > 0$ such that $B(x,r) \subset E$.*

(2) E is called **open** *if every point of E is an interior point.*

(3) E is called **closed** *if $\mathbb{R}^n \setminus E$ is open. Equivalently, E is closed if and only if for any sequence (x_k) in E such that $x_k \to x$, we have $x \in E$.*

(4) A point $x \in \mathbb{R}^2$ is called a **boundary point** *of E if for every $r > 0$, $B(x,r) \cap E \neq \emptyset$ and $B(x,r) \cap (\mathbb{R}^n \setminus E) \neq \emptyset$. The set of boundary points of E is called the boundary of E and is denoted by ∂E.*

The next theorem shows that convergence in $\mathbb{R}^n$ can be described in terms of coordinate-wise convergence.

Theorem 7. *Let $(x_k) = ((x_{k1}, \ldots, x_{kn}))$ be a sequence in $\mathbb{R}^n$, and let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Then $x_k \to x$ as $k \to \infty$ if and only if for each $j = 1, \ldots, n$, $x_{kj} \to x_j$ as $k \to \infty$.*

Proof. Observe that for each $j = 1, \ldots, n$, we have

$$|x_{kj} - x_j| \leq ||x_k - x|| \leq \sum_{i=1}^{n} |x_{ki} - x_i|.$$

If $x_k \to x$ as $k \to \infty$, i.e., $||x_k - x|| \to 0$ as $k \to \infty$, then from the above inequality, we have $|x_{kj} - x_j| \to 0$ as $k \to \infty$ for each $j = 1, \ldots, n$.

Conversely, if for each $j = 1, \ldots, n$, we have $|x_{kj} - x_j| \to 0$ as $k \to \infty$, then $\sum_{i=1}^{n} |x_{ki} - x_i| \to 0$ $k \to \infty$. This implies that $||x_k - x|| \to 0$ $k \to \infty$. □

Examples.

(1) $((e^k, k^2))$ is divergent since (k^2) is divergent.
(2) $(\frac{1}{k}, \frac{1}{k!}) \to (0, 0)$.
(3) $((\cos\frac{\pi}{k}, \sin\frac{\pi}{k}))$ is divergent because both sequences $(\cos\frac{\pi}{k})$ and $(\sin\frac{\pi}{k})$ diverge.
(4) $((\frac{\log k}{k}, \frac{k^2}{e^k}, (-1)^k))$ diverges.
(5) $((\sqrt{k+1} - \sqrt{k}, \frac{1}{e}, k\pi))$ diverges.
(6) $(1, -1, \sqrt{2}) \to (1, -1, \sqrt{2})$.

Examples.

(1) The intervals (a, b), $[a, b]$ and $(a, b]$ is open, closed, neither open nor closed respectively.
(2) The set $E_1 = \{(x, y) \in \mathbb{R}^2 : y = 0\}$ is closed but not open.
(3) The set $E_2 = \{(x, y, z) \in \mathbb{R}^3 : ||(x, y, z)|| < 1\}$ is open but not closed.
(4) The set $E_3 = \mathbb{Q} \times \mathbb{Q}$ is neither open nor closed.

Proposition 8. *Every convergent sequence in $\mathbb{R}^n$ is bounded.*

Proof. Exercise. □

Definition 9. *Let $(x_k) = ((x_{k1}, \ldots, x_{kn}))$ be a sequence in $\mathbb{R}^n$. If $k_1, k_2, \ldots$ are positive integers such that $k_j < k_{j+1}$ for each $j \in \mathbb{N}$, then the sequence (x_{k_j}) is called a subsequence of (x_k).*

Exercise 10. *Give an example of a subsequence for some of the sequences given in this lecture.*

Theorem 11 (Bolzano-Weierstrass)**.** *Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.*