IMPROPER RIEMANN INTEGRALS

In lecture 16, we considered the Riemann integral of a bounded function defined on a closed and bounded interval. In this lecture, we will extend the theory of integration to functions defined on a semi-infinite interval or a doubly infinite interval, and also to unbounded functions defined on bounded or unbounded intervals.

Improper Integral of the First Kind

Let $a \in \mathbb{R}$, and let Let $f : [a, \infty) \to \mathbb{R}$ be a function. Suppose that f is (Riemann) integrable on $[a, x]$ for every $x \geq a$.

Definition 1. *If* $\lim_{x\to\infty} \int_a^x f(t)dt = \ell$, *for some* $\ell \in \mathbb{R}$, *then we say that the* **improper integral** *(of the first kind)* $\int_a^\infty f(t)dt$ **converges to** ℓ, *and we write* $\int_a^\infty f(t)dt = \ell$. *Otherwise, we say that the improper integral* $\int_a^\infty f(t)dt$ *diverges.*

If $\int_a^x f(t)dt \to \infty$ *or* $\int_a^x f(t)dt \to -\infty$ *as* $x \to \infty$, *then we say that improper integral diverges to* ∞ *or to* $-\infty$, *as the case may be.*

Remark 2. *Observe that there is a remarkable analogy between the definitions of an infinite series* $\sum_{n=1}^\infty x_n$ *and an infinite integral* $\int_a^\infty f(t)dt$. *The sequence* (x_n) *corresponds to the function* $f : [a, \infty) \to \mathbb{R}$, *and the partial sum* $S_n = x_1 + x_2 + \cdots + x_n$ *corresponds to the 'partial integral'* $F(x) = \int_a^x f(t)dt$, *where* $x \in [a, \infty)$.

Examples.

(1) Let $p \in \mathbb{R}$ and consider the improper integral $\int_1^\infty \frac{1}{t^p}\, dt$. For any $x \in [1, \infty)$, we have

$$\int_1^x \frac{1}{t^p}\, dt = \begin{cases} \frac{x^{1-p}-1}{1-p}, & \text{if } p \neq 1, \\ \log x, & \text{if } p = 1. \end{cases}$$

It follows that if $p > 1$, then $\int_1^\infty \frac{1}{t^p}\, dt$ converges to $\frac{1}{p-1}$, and if $p \leq 1$, then $\int_1^\infty \frac{1}{t^p}\, dt$ diverges to ∞.

(2) The improper integral $\int_0^\infty te^{-t^2}dt$ converges to $\frac{1}{2}$. In this case,

$$\int_0^x te^{-t^2}dt = \frac{1}{2}\int_0^{x^2} e^{-s}ds = \frac{1}{2}(1 - e^{-x^2}) \to \frac{1}{2} \text{ as } x \to \infty.$$

(3) The integral $\int_0^\infty \cos t\, dt$ diverges, because $\int_0^x \cos t\, dt = \sin x$ for all $x \in \mathbb{R}$ and $\lim_{x\to\infty} \sin x$ does not exist.

Convergence Tests for Improper Integrals

Definition 3. *An improper integral $\int_a^\infty f(t)dt$ is said to be absolutely convergent if the improper integral $\int_a^\infty |f(t)|dt$ is convergent.*

The following result is anticipated.

Proposition 4. *An absolutely convergent improper integral is convergent.*

Theorem 5 (Necessary and Sufficient Condition)**.** *Let $f : [a,\infty) \to \mathbb{R}$ be a nonnegative function. Then $\int_a^\infty f(t)dt$ is convergent if and only if the function $F : [a,\infty) \to \mathbb{R}$ defined by $F(x) = \int_a^x f(t)dt$ is bounded above, i.e., $\exists\ M > 0$ such that $|\int_a^x f(t)dt| \le M$ for all $x \ge a$. In this case we have,*

$$\int_a^\infty f(t)dt = \sup\{F(x) : x \in [a,\infty)\}.$$

If F is not bounded above, then $\int_a^\infty f(t)dt$ diverges to ∞.

Example.

(1) $\int_0^\infty \frac{1+\sin t}{1+t^2}dt$ is convergent. Since, $\frac{1+\sin t}{1+t^2} \ge 0$ and

$$F(x) = \int_0^x \frac{1+\sin t}{1+t^2}dt \le \int_0^x \frac{2}{1+t^2}dt = 2\tan^{-1} x \le \pi\ \forall\ x \in [0,\infty).$$

(2) $\int_0^\infty \frac{2+\cos t}{t}dt$ diverges to ∞. We have $\frac{2+\cos t}{t} \ge 0$ and

$$F(x) = \int_0^x \frac{2+\cos t}{t}dt \ge \int_0^x \frac{1}{t}dt = \log x \to \infty \text{ as } x \to \infty.$$

Theorem 6 (Comparison Test)**.** *Suppose $a \in \mathbb{R}$ and $f, g : [a,\infty) \to \mathbb{R}$ are such that both f and g are integrable on $[a, x]$ for every $x \ge a$ and $|f| \le g$. If $\int_a^\infty g(t)dt$ is convergent, then $\int_a^\infty f(t)dt$ is absolutely convergent and*

$$\left|\int_a^\infty f(t)dt\right| \le \int_a^\infty g(t)dt.$$

Examples.

(1) $\int_1^\infty \frac{\cos^2 t}{t^2}\ dt$ converges, because, $\frac{\cos^2 t}{t^2} \le \frac{1}{t^2}$.

(2) $\int_1^\infty \frac{(2+sint)}{t}\ dt$ diverges, because, $\frac{1}{t} \le \frac{(2+sint)}{t}$.

Theorem 7 (Limit Comparison Test (LCT))**.** *Suppose $a \in \mathbb{R}$ and $f, g : [a,\infty) \to \mathbb{R}$ are such that both f and g are integrable on $[a, x]$ for every $x \ge a$ and $g(t) \ne 0$ for all large t. Assume $\lim_{t\to\infty} \frac{f(t)}{g(t)} = \ell$, where $\ell \in \mathbb{R}$ or $\ell = \pm\infty$.*

(1) If $g(t) > 0$ for all large t, $\int_a^\infty g(t)dt$ is convergent, and $\ell \in \mathbb{R}$, then $\int_a^\infty f(t)dt$ is absolutely convergent.

(2) If $f(t) > 0$ for all large t, $\int_a^\infty f(t)dt$ is convergent, and $\ell \ne 0$, then $\int_a^\infty g(t)dt$ is absolutely convergent.

Examples.

(1) The integral $\int_1^\infty \sin\frac{1}{t}dt$ diverges, because $\frac{sin\frac{1}{t}}{\frac{1}{t}} \to 1$ as $t \to \infty$.

(2) For $p \in \mathbb{R}$, $\int_1^\infty e^{-t}t^p dt$ converge, because $\frac{e^{-t}t^p}{t^{-2}} \to 0$ as $t \to \infty$.

Theorem 8 (Dirichlet test)**.** *Let $f, g : [a, \infty) \to \mathbb{R}$ be such that*

(1) f is decreasing and $f(t) \to 0$ as $t \to \infty$, and

(2) g is continuous and $\int_a^x g(t)dt$ is bounded.

Then $\int_a^\infty f(t)g(t)dt$ converges.

Example. The Integrals $\int_\pi^\infty \frac{\sin t}{t}dt$ and $\int_\pi^\infty \frac{\cos t}{t}dt$ are convergent.

Related Integrals

Improper integrals of the form $\int_{-\infty}^b f(t)dt$ are defined similarly.

Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is integrable on $[a, b]$ for all $a, b \in \mathbb{R}$ with $a \le b$.

Definition 9 (Doubly infinite integral)**.** *We say that $\int_{-\infty}^\infty f(t)dt$ is convergent if both $\int_{-\infty}^0 f(t)dt$ and $\int_0^\infty f(t)dt$ are convergent, i.e., if the limits*

$$\lim_{x\to-\infty}\int_x^0 f(t)dt \text{ and } \lim_{x\to\infty}\int_0^x f(t)dt$$

both exists. In this case,

$$\int_{-\infty}^\infty f(t)dt = \lim_{x\to-\infty}\int_x^0 f(t)dt + \lim_{x\to\infty}\int_0^x f(t)dt.$$

If any one of these limits does not exist, we say that $\int_{-\infty}^\infty f(t)dt$ is divergent.

Definition 10 (Cauchy principal value)**.** *If the limit*

$$\lim_{x\to\infty}\int_{-x}^x f(t)dt$$

exists, then it is called the **Cauchy principal value** *of the integral of f over $\mathbb{R}$.*

Remark 11. *(1) If $\int_{-\infty}^\infty f(t)dt$ is convergent, then since*

$$\int_{-x}^x f(t)dt = \int_{-x}^0 f(t)dt + \int_0^x f(t)dt \text{ for all } x \ge 0,$$

the Cauchy principal value of the integral of f on $\mathbb{R}$ exists and is equal to $\int_{-\infty}^\infty f(t)dt$.

(2) The Cauchy principal value of the integral of f on $\mathbb{R}$ may exist even when $\int_{-\infty}^\infty f(t)dt$ is divergent. For example, consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(t) = \sin t$. For every $x \ge 0$, we have

$$\int_0^x \sin t dt = 1 - \cos t = -\int_{-x}^0 \sin t \text{ and so } \int_{-x}^x \sin t dt = 0.$$

Hence, $\lim_{x\to\infty}\int_{-x}^x f(t)dt = 0$ but neither of the two limits $\lim_{x\to\infty}\int_{-x}^0 f(t)dt$ $\lim_{x\to\infty}\int_0^x f(t)$ exists.

(3) If $f : \mathbb{R} \to \mathbb{R}$ is a nonnegative function and Cauchy principal value of the integral of f on $\mathbb{R}$ exists, then $\int_{-\infty}^{\infty} f(t)dt$ is convergent. Try to prove this!

Improper Integral of the Second Kind

Let $a, b \in \mathbb{R}$ with $a < b$, and $f : (a, b] \to \mathbb{R}$ such that f is unbounded on $(a, b]$ but integrable on $[x, b]$ for each $x \in (a, b]$. If $\lim_{x\to a^+} \int_x^b f(t)dt = \ell$ for some $\ell \in \mathbb{R}$, then we say that the improper integral (of the second kind) $\int_a^b f(t)dt$ converges to ℓ, and we write $\int_a^b f(t)dt = \ell$. If $\int_a^b f(t)dt$ is not convergent, then it is said to be divergent.

Example. The improper integral $\int_0^1 \frac{1}{t^p}\, dt$ converges for $p < 1$ and diverges for $p \geq 1$.

Comparison test and limit comparison test for improper integral of the second kind are analogous to those of the first kind. If an improper integral is a combination of both first and second kind then one defines the convergence similar to that of the improper integral of the kind $\int_{-\infty}^{\infty} f(t)dt$.

Problem 12. *Determine the values of p for which $\int_0^{\infty} \frac{1-e^{-x}}{x^p}\, dt$ converges.*

Solution. Let $I_1 = \int_0^1 \frac{1-e^{-x}}{x^p} dx$ and $I_2 = \int_1^{\infty} \frac{1-e^{-x}}{x^p} dx$. We have to determine the values of p for which the integrals I_1 and I_2 converge.

Since $\lim_{x\to 0} \frac{1-e^{-x}}{x} = 1$, by LCT with $\frac{1}{x^{p-1}}$, we see that I_1 is convergent if and only if $p - 1 < 1$, i.e., $p < 2$.

Similarly, I_2 is convergent (by applying LCT with $\frac{1}{x^p}$) if and only if $p > 1$. Therefore, $\int_0^{\infty} \frac{1-e^{-x}}{x^p}\, dt$ converges if and only if $1 < p < 2$. □

Problem 13. *Prove that $\int_1^{\infty} \frac{\sin x}{x^p} dx$ converges but not absolutely for $0 < p \leq 1$.*

Solution. Let $0 < p \leq 1$. By Dirichlet's Test, the integral converges. We claim that $\int_1^{\infty} \frac{|\sin x|}{x^p} dx$ does not converge. Since, $|\sin x| \geq \sin^2 x$ we see that $|\frac{\sin x}{x^p}| \geq \frac{\sin^2 x}{x^p} = \frac{1-\cos 2x}{2x^p}$. By Dirichlet's Test, $\int_1^{\infty} \frac{\cos 2x}{2x^p} dx$ converges for all $p > 0$. But $\int_1^{\infty} \frac{1}{2x^p} dx$ diverges for $p \leq 1$. Hence,$\int_1^{\infty} |\frac{\sin x}{x^p}| dx$ does not converge. □