THE FUNDAMENTAL THEOREMS OF CALCULUS

Differentiation and integration are the two most important processes in calculus and analysis. Differentiation is a local process, i.e., the value of the derivative at a point depends only on the values of the function in a small interval about that point. On the other hand, integration is a global process in the sense that the integral of a function depends on the values of the function on the entire interval. Further, these processes are defined in entirely different manners and there is no apparent connection between them. From a geometric point of view, differentiation corresponds to finding slopes of tangents to curves, while integration corresponds to finding areas under curves. By looking at the definitions, there seems to be no reason for these two geometric processes to be intimately related.

In this lecture, we will study one of the most important results in the theory of integration, namely, the fundamental theorems of calculus, (in short, FTC). These theorems establish the validity of the computation of integrals via Newtonian calculus, as learned in school. In some sense, they justify the school way of defining integration as finding an anti-derivative.

But before we start, let us look at some properties of the integral.

Algebraic and Order Properties of the Integral

First we consider how Riemann integration behaves with respect to the algebraic operations on functions.

Theorem 1. *Let $f, g : [a, b] \to \mathbb{R}$ be integrable functions and $c \in \mathbb{R}$. Then:*

(1) cf is integrable and $\int_a^b (cf)(x)\mathrm{d}x = c\int_a^b f(x)\mathrm{d}x$.
(2) $f + g$ is integrable and $\int_a^b (f+g)(x)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x + \int_a^b g(x)\mathrm{d}x$.
(3) Let $\mathcal{R}([a, b])$ denote the set of integrable functions on $[a, b]$. Then $\mathcal{R}([a, b])$ is a vector space. Moreover, the map $T : \mathcal{R}([a, b]) \to \mathbb{R}$ defined as $T(f) = \int_a^b f(x)\mathrm{d}x$ is linear.

The next theorem shows how Riemann integration behaves with respect to the order relation on functions.

Theorem 2. *Let $f, g : [a, b] \to \mathbb{R}$ be integrable.*

(1) (Monotonicity of the Integral). If $f \le g$, then $\int_a^b f(x)\mathrm{d}x \le \int_a^b g(x)\mathrm{d}x$.

(2) (Basic Estimate for Integrals). The function $|f|$ is integrable and $\left|\int_a^b f(x)\mathrm{d}x\right| \leq \int_a^b |f|(x)\mathrm{d}x$.

Proposition 3 (Domain Additivity of Riemann Integrals)**.** *Let $f : [a, b] \to \mathbb{R}$ be a bounded function and let $c \in (a, b)$. Then f is integrable on $[a, b]$ if and only if f is integrable on $[a, c]$ and on $[c, b]$. In this case,*

$$\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x.$$

Definition 4. *Let $f : [a, b] \to \mathbb{R}$ be a integrable. We define*

$$\int_b^a f(x)\mathrm{d}x = -\int_a^b f(x)\mathrm{d}x \text{ and } \int_s^s f(x)\mathrm{d}x = 0.$$

FTC

Now, we state the two fundamental theorems of calculus. The first one involves the derivative of the integral, and the second one involves the integral of the derivative.

Theorem 5 (First Fundamental Theorem of Calculus)**.** *Let $f : [a, b] \to \mathbb{R}$ be integrable. Let $F : [a, b] \to \mathbb{R}$ be the function $F(x) = \int_a^x f(t)\mathrm{d}t$. Then F is continuous. Furthermore, if f is continuous at $x_0 \in [a, b]$, then F is differentiable at x_0, and $F'(x_0) = f(x_0)$.*

Informally, the first fundamental theorem of calculus asserts that $\left(\int_a^x f\right)'(x) = f(x)$ given a certain number of assumptions on f. Roughly, this means that the derivative of an integral recovers the original function. The next theorem shows the reverse, i.e., the integral of a derivative recovers the original function.

Definition 6 (Antiderivatives)**.** *$f : [a, b] \to \mathbb{R}$ be a function. We say that $F : [a, b] \to \mathbb{R}$ is an antiderivative or a primitive of f if F is differentiable on $[a, b]$ and $F'(x) = f(x)$ for all $x \in [a, b]$.*

Theorem 7 (Second Fundamental Theorem of Calculus)**.** *Let $f : [a, b] \to \mathbb{R}$ be integrable. If $F : [a, b] \to \mathbb{R}$ is an antiderivative of f, then*

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a).$$

Remark 8. *This justifies what we leaned in school the integral being anti-derivative. That is, to find $\int_a^b f(x)\mathrm{d}x$, we find a function F such that $F' = f$.*

Riemann Sums

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. The calculation $U(P, f)$ and $L(P, f)$ for a given partition P, involves finding the absolute maxima and minima of f over several subintervals of [a, b]. This task is not an easy task. In this section we discuss another easier approach.

Definition 9. *Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$, and let $s_i \in [x_{i-1}, x_i]$ for $i = 1, \ldots, n$. Then s_i's are called tags. Let $\boldsymbol{s} = \{s_i : i = 1, \ldots, n\}$ be the set of tags. The pair $(P, \boldsymbol{s})$ is called a tagged partition of $[a, b]$. The sum defined by*

$$S(P, f) = \sum_{i=1}^{n} f(s_i)(x_i - x_{i-1})$$

is called a Riemann sum for f corresponding to P.

Note that $S(P, f)$ depend on P and f, and also on $\boldsymbol{s}$.

Definition 10. *Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. We define the mesh or the norm of P as $||P|| = \max\{x_i - x_{i-1} : i = 1, \ldots, n\}$.*

Theorem 11 (Theorem of Darboux)**.** *Let $f : [a, b] \to \mathbb{R}$ be a bounded function.*

(1) If f is integrable, then given any $\varepsilon > 0$, there is $\delta > 0$ such that for every partition P of $[a, b]$ with $||P|| < \delta$, we have

$$\left| S(P, f) - \int_a^b f(x)\mathrm{d}x \right| < \varepsilon.$$

(2) Conversely, assume that there is $r \in \mathbb{R}$ satisfying the following condition: Given $\varepsilon > 0$, there is a partition P of $[a, b]$ such that $|S(P, f) - r| < \varepsilon$, then f is integrable and its Riemann integral is r.

An immediate consequence of the above result is the following.

Corollary 12. *If $f : [a, b] \to \mathbb{R}$ is integrable and if (P_n) is a sequence of partitions of $[a, b]$ such that $||P_n|| \to 0$, then*

$$L(P_n, f) \to \int_a^b f(x)\mathrm{d}x,\ U(P_n, f) \to \int_a^b f(x)\mathrm{d}x \text{ and } S(P_n, f) \to \int_a^b f(x)\mathrm{d}x.$$

Remark 13. *Note that the only requirement in the above corollary is that $||P_n|| \to 0$. The actual partition points and the points in the subintervals at which f is evaluated can be chosen with sole regard to the convenience of summation.*

Application of Riemann sum in finding limit of a sequence

We can determine limits of some sequences by expressing the nth term as a Riemann sum for a suitable function. Let us consider the following example.

Example 14. *Find $\lim_{n\to\infty} x_n$, where $x_n = \frac{n}{1+n^2} + \frac{n}{2^2+n^2} + \cdots + \frac{n}{n^2+n^2}$.*

Solution. Note that

$$\begin{aligned} x_n &= \frac{n}{1+n^2} + \frac{n}{2^2+n^2} + \cdots + \frac{n}{n^2+n^2} \\ &= \frac{1}{n}\left(\frac{n^2}{1+n^2} + \frac{n^2}{2^2+n^2} + \cdots + \frac{n^2}{n^2+n^2}\right) \\ &= \frac{1}{n}\left(\frac{1}{(\frac{1}{n})^2+1} + \frac{1}{(\frac{2}{n})^2+1} + \cdots + \frac{1}{(\frac{n}{n})^2+1}\right) \\ &= S(P_n, f), \end{aligned}$$

where

$$f(x) = \frac{1}{x^2+1},\ P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n} = 1\right\} \text{ and } s_i = \frac{i}{n},\ i = 1, \ldots, n.$$

Since f is continuous on $[0,1]$, it is integrable. Moreover, $||P_n|| = \frac{1}{n} \to 0$. Therefore, by Corollary 12,

$$x_n = S(P_n, f) \to \int_0^1 \frac{1}{x^2+1}\mathrm{d}x = \tan^{-1} x|_0^1 = \frac{\pi}{4}.$$