RIEMANN INTEGRATION

The Riemann Integral

The notion of integration was developed much earlier than differentiation. The main idea of integration is to assign a real number A, called the "area", to the region bounded by the curves $x = a$, $x = b$, $y = 0$, and $y = f(x)$. To proceed formally, we introduce the following concept.

Definition 1. *By a partition P of $[a, b]$ we mean a finite ordered set $\{x_0, x_1, \ldots, x_n\}$ of points in $[a, b]$ such that*

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

Examples.

(1) The simplest partition of $[a, b]$ is given by $P_1 = \{a, b\}$.

(2) For $n \in \mathbb{N}$, let $P_n = \{x_0, x_1, \ldots, x_n\}$, where

$$x_i = a + \frac{i(b-a)}{n}, \quad \text{for } i = 0, 1, \ldots, n.$$

Then P_n is a partition that subdivides the interval $[a, b]$ into n subintervals, each of length $(b - a)/n$. What happens when n becomes larger?

Let $f : [a, b] \to \mathbb{R}$ be a bounded function, and $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. Define

$$m = \inf\{f(x) : x \in [a, b]\} \text{ and } M = \sup\{f(x) : x \in [a, b]\}.$$

For $i = 1, 2, \ldots, n$,

$$m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\} \text{ and } M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}.$$

Clearly,

$$m \leq m_i \leq M_i \leq M \text{ for all } i = 1, 2, \ldots, n. \text{ (Prove!)}$$

Definition 2. *Given a bounded function $f : [a, b] \to \mathbb{R}$ and a partition $P : \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$, the lower Riemann[1] sum of f with respect to the partition P is defined as*

$$L(P, f) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}).$$

[1]Bernhard Riemann was a German mathematician who made contributions to analysis, number theory, and differential geometry. A work which Riemann did in 1859 is referred to as the Riemann hypothesis. Anyone who solves the Riemann hypothesis will earn a **million dollar**!

The upper Riemann sum of f with respect to the partition P is defined as

$$U(P,f)=\sum_{i=1}^{n} M_i(x_i-x_{i-1}).$$

Lemma 3. *Let $f:[a,b]:\to\mathbb{R}$ be a bounded function. Then for any partition P of $[a,b]$, we have*

$$m(b-a)\le L(P,f)\le U(P,f)\le M(b-a).$$

Proof. Exercise. □

Definition 4. *The upper and lower Riemann integral of f over $[a,b]$ is respectively defined as*

$$L(f)=\underline{\int_a^b} f(x)\mathrm{d}x=\sup\{L(P,f): P \textit{ is a partition of}[a,b]\},$$

and

$$U(f)=\overline{\int_a^b} f(x)\mathrm{d}x=\inf\{L(P,f): P \textit{ is a partition of}[a,b]\}.$$

Does these two numbers exist? If the upper and lower Riemann integrals are equal, we say that f is Riemann integrable or simply integrable. In this case, the common value of $L(f)=U(f)$ is called the Riemann integral of f (on $[a,b]$ and is denoted by

$$\int_a^b f(x)\mathrm{d}x \textit{ or simply } \int_a^b f.$$

Examples.

(1) Let $f:[a,b]:\to\mathbb{R}$ be a constant function with $f(x)=c$ for all x. Let P be any partition of $[a,b]$. Then $m_i=M_i=c$ for all i and $L(P,f)=U(P,f)=c(b-a)$. Thus, $\underline{\int_a^b} f(x)\mathrm{d}x=c(b-a)=\overline{\int_a^b} f(x)\mathrm{d}x$, and hence f is Riemann integrable. This implies that $\int_a^b f(x)\mathrm{d}x=c(b-a)$.

(2) Let $\lambda>1$ be a real number. Let $f:[0,1]:\to\mathbb{R}$ be defined by

$$f(x)=\begin{cases}1, & \text{if } 0\le x<1,\\ \lambda, & \text{if } x=1.\end{cases}$$

Let us find the upper and lower Riemann integrals. Let $P=\{x_0,\dots,x_n\}$ be a partition of $[0,1]$. We observe that $m_i=1$ for $i=1,\dots,n$. Moreover, $M_i=1$ for $i=1,\dots,n-1$ and $M_n=\lambda$. It follows that

$$L(P,f)=1 \text{ and } \underline{\int_0^1} f(x)\mathrm{d}x=1.$$

On the other hand, $U(P,f)=x_{n-1}+\lambda(1-x_{n-1})$ and

$$\overline{\int_0^1} f(x)\mathrm{d}x=\inf\{U(P,f): P \text{ is a partition of } [0,1]\}=\inf(1,\lambda]=1 \text{ (Verify!)}.$$

This shows that f is integrable and $\int_0^1 f(x)\mathrm{d}x=1$.

(3) Consider the Dirichlet's function, i.e., the function defined by $f : [0,1] \to \mathbb{R}$ defined by $f(x) = 1$ if $x \in \mathbb{Q}$ and 0 otherwise. Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[0,1]$. In any subinterval $[x_{i-1}, x_i]$, there exist some rational numbers and irrational numbers. Hence, $m_i = 0$ and $M_i = 1$. So, $L(P, f) = 0$ and $U(P, f) = 1$. It follows that $\underline{\int_a^b} f(x)\mathrm{d}x = 0$ and $\overline{\int_a^b} f(x)\mathrm{d}x = 1$. Thus, f is not integrable on $[0,1]$.

Integrable Functions

Definition 5. *Given a partition P of $[a,b]$, we say that a partition P^* of $[a,b]$ is a refinement of P if $P \subset P^*$. Given partitions P_1 and P_2 of $[a,b]$, the partition $P^* = P_1 \cup P_2$ is called the common refinement of P_1 and P_2.*

Proposition 6. *Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Then we have the following:*

(1) If P is partition of $[a,b]$, and P^* is a refinement of P, then

$$L(P, f) \leq L(P^*, f) \text{ and } U(P^*, f) \leq U(P, f),$$

and consequently,

$$U(P^*, f) - L(P^*, f) \leq U(P, f) - L(P, f).$$

(2) If P_1 and P_2 are partitions of $[a,b]$, then $L(P_1, f) \leq U(P_2, f)$.

(3) $\underline{\int_a^b} f(x)\mathrm{d}x \leq \overline{\int_a^b} f(x)\mathrm{d}x$

In the following result we present a necessary and sufficient condition for the existence of the integral of a bounded function.

Theorem 7 (Riemann's criterion for integrability)**.** *Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Then f is integrable if and only if for every $\varepsilon > 0$, there is a partition P_ε of $[a,b]$ such that*

$$U(P_\varepsilon, f) - L(P_\varepsilon, f) < \varepsilon.$$

The proof of the following corollary is immediate from the previous theorem.

Corollary 8. *Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Suppose (P_n) is a sequence of partitions of $[a,b]$ such that $U(P_n, f) - L(P_n, f) \to 0$. Then f is integrable.*

Example. Let $f(x) = x^2$ on $[0,1]$. Let $\varepsilon > 0$ be given. Choose a partition P such that $\max\{x_i - x_{i-1} : 1 \leq i \leq n\} < \varepsilon/2$. Since f is increasing, we have

$$m_i = f(x_{i-1}) = x_{i-1}^2 \text{ and } M_i = f(x_i) = x_i^2.$$

This implies that

$$U(P,f)-L(P,f)=\sum_{i=1}^{n}x_i^2(x_i-x_{i-1})-\sum_{i=1}^{n}x_{i-1}^2(x_i-x_{i-1})$$
$$=\sum_{i=1}^{n}(x_i-x_{i-1})(x_i+x_{i-1})((x_i-x_{i-1})$$
$$<\sum_{i=1}^{n}\left[\left(\frac{\varepsilon}{2}\right)\times 2\right](x_i-x_{i-1}),\ \text{since } 0\le, x_{i-1},x_i\le 1$$
$$=\varepsilon\sum_{i=1}^{n}(x_i-x_{i-1})=\varepsilon.$$

Hence, f is integrable.

We will apply the Riemann's criterion for integrability to prove the following theorem.

Theorem 9. *Let $f:[a,b]\to\mathbb{R}$ be a function.*

(1) If f is monotone, then it is integrable.
(2) If f is continuous, then it is integrable.

Proof. We proof part (1). The the proof of part (2) is left to the reader.

Suppose f is monotonically increasing (the proof is similar in the other case). Choose a partition P such that $x_i-x_{i-1}=\frac{b-a}{n}$ for each i. Then $M_i=f(x_i)$ and $m_i=f(x_{i-1})$. Thus, for large n, we have

$$U(P,f)-L(P,f)=\frac{b-a}{n}\sum_{i=1}^{n}(f(x_i)-f(x_{i-1}))=\frac{b-a}{n}\sum_{i=1}^{n}(f(b)-f(a))<\varepsilon.$$

Hence, f is integrable. □

We end this lecture with the following problem. We encourage students to understand and verify each step.

Problem 10. *Let $f:[0,1]\longrightarrow\mathbb{R}$ be defined as*

$$f(x)=\begin{cases}x, & x \text{ is rational}\\ 0, & x \text{ is irrational.}\end{cases}$$

Evaluate the upper and lower integrals of f and show that f is not integrable.

Solution. Let $P=\{x_0,x_1,\ldots,x_n\}$ be any partition of $[0,1]$. Since there exists an irrational number in each subinterval $[x_{i-1},x_i]$, $L(P,f)=0$, and hence $\underline{\int_0^1} f(x)\mathrm{d}x=0$.

Now,

$$\begin{aligned}
U(P,f) &= \sum_{i=1}^{n} x_i(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} x_{i-1}x_i \\
&\geq \sum_{i=1}^{n} x_i^2 - \frac{1}{2}\sum_{i=1}^{n}(x_{i-1}^2 + x_i^2) \text{ (Using AM-GM inequality)} \\
&= \frac{1}{2}\sum_{i=1}^{n}(x_{i-1}^2 - x_i^2) = \frac{1}{2}. \\
\implies \overline{\int_0^1} f(x)\mathrm{d}x &\geq \frac{1}{2}.
\end{aligned}$$

For each $n \in \mathbb{N}$, consider $P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n}, \frac{n}{n} = 1\right\}$.

Then

$$\begin{aligned}
U(P_n, f) &= \frac{n+1}{2n} = \frac{1}{2} + \frac{1}{2n} \\
\implies &\quad \inf\left\{U(P_n, f) : n \in \mathbb{N}\right\} = \frac{1}{2} \\
\implies &\quad \overline{\int_0^1} f(x)\mathrm{d}x \leq \frac{1}{2}.
\end{aligned}$$

Therefore, $\overline{\int_0^1} f(x)\mathrm{d}x = \frac{1}{2}$. □