POWER SERIES

In this lecture we will define a class of functions that are very important in Calculus. So far, you have encountered many transcendental functions[1] such as trigonometric, logarithmic and exponential functions. You also use to write such functions in the form of series since your school days. This series is known as Taylor series. We will see that many classical functions admits a Taylor series and we will study its convergence.

Definition 1. *A power series around a is an expression of the form $\sum_{n=0}^{\infty} a_n(x-a)^n$ where $a_n, a, x \in \mathbb{R}$.*

If we let $\tilde{x} = x - a$, then the power series around a can be reduced to a power series around 0. We are interested in finding $x \in \mathbb{R}$ for which the series $\sum_{n=0}^{\infty} a_n x^n$ is convergent.

Consider the following three power series.

(1) $\sum_{n=1}^{\infty} n^n x^n$. This series converges only at $x = 0$. Indeed, if $x \neq 0$, we can choose $N \in \mathbb{N}$ such that $\frac{1}{N} < |x|$. Thus, for all $n \geq N$, we have $|(nx)^n| > 1$ and hence the series is divergent.

(2) $\sum_{n=0}^{\infty} x^n$. We have seen that this series converges absolutely for $|x| < 1$.

(3) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. Using ratio test we can show that this series converges absolutely for all $x \in \mathbb{R}$.

The general phenomenon is given in the following lemma.

Lemma 2 (Abel's Lemma). *Let $x_0 \in \mathbb{R}$ such that $\{a_n x_0^n : n \in \mathbb{N}\}$ is bounded. Then $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent for every $x \in \mathbb{R}$ with $|x| < |x_0|$. In particular, if $\sum_{n=0}^{\infty} a_n x_0^n$ is convergent, then $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent for all x such that $|x| < |x_0|$.*

[1] A function $f : D(\subseteq \mathbb{R}) \to \mathbb{R}$ is said to be an algebraic function if $y = f(x)$ satisfies an equation whose coefficients are polynomials, i.e., $p_n(x)y^n + p_{n-1}(x)y^{n-1} + \cdots + p_1(x)y + p_0(x) = 0$ for $x \in D$, where $n \in \mathbb{N}$ and $p_0(x), p_1(x), \ldots, p_n(x)$ are polynomials such that $p_n(x)$ is nonzero. For example, $y = f(x) = \sqrt[n]{x}$ is an algebraic function since y satisfies the equation $y^n - x = 0$ for $x \in [0, \infty)$. A real-valued function that is not algebraic is called a transcendental function. A real number α is called an algebraic number if it satisfies a nonzero polynomial with integer coefficients. Numbers that are not algebraic are called transcendental numbers. For example, $\sqrt{2}, \sqrt{3}, \sqrt[5]{7}$ and $\sqrt{2} + \sqrt{3}$ are algebraic numbers. Any rational number is algebraic. The numbers e and π are transcendental. For more details, see [1] and [2].

Theorem 3. *A power series $\sum_{n=0}^{\infty} a_n x^n$ is either absolutely convergent for all $x \in \mathbb{R}$ or there is a unique nonnegative real number R such that the series is absolutely convergent for all x with $|x| < R$ and is divergent for $|x| > R$.*

Definition 4. *We say that the* **radius of convergence** *of a power series is ∞ if the power series is absolutely convergent for all $x \in \mathbb{R}$; otherwise, it is defined to be the unique nonnegative real number R such that the power series is absolutely convergent for all x with $|x| < R$ and is divergent for $|x| > R$.*

Let $S = \{x \in \mathbb{R} : \sum_{n=0}^{\infty} a_n x^n \text{ is convergent}\}$. Then $S \neq \emptyset$ (Why?).

Exercise. The radius of convergence and the set S for various power series are given below. Verify?

Power Series	Radius of convergence	S
$\sum_{n=1}^{\infty} n^n x^n$	0	$\{0\}$
$\sum_{n=1}^{\infty} x^n/n!$	∞	$(-\infty, \infty)$
$\sum_{n=0}^{\infty} x^n$	1	$(-1, 1)$
$\sum_{n=0}^{\infty} x^n/n^2$	1	$[-1, 1]$
$\sum_{n=0}^{\infty} x^n/n$	1	$[-1, 1)$
$\sum_{n=0}^{\infty}(-1)^n x^n/n$	1	$(-1, 1]$

The following result is useful in calculating the radius of convergence of a power series.

Proposition 5. *Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series and R be its radius of convergence.*

(1) If $|a_n|^{1/n} \to \ell$, then

$$R = \begin{cases} 0 & \text{if } \ell = \infty, \\ \infty & \text{if } \ell = 0, \\ 1/\ell & \text{if } \ell > 0. \end{cases}$$

(2) If $a_n \neq 0$ eventually and $|\frac{a_{n+1}}{a_n}| \to \ell$, then the above conclusion holds.

Taylor and Maclaurin Series

Let $f : [a, b] \to \mathbb{R}$ be an infinitely differentiable function and $c \in (a, b)$. In one of previous lectures, we have defined the Taylor polynomial of order n generated by f at c as the polynomial

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k.$$

The power series

$$f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\cdots+\frac{f^{(n)}(c)}{n!}(x-c)^n+\cdots=\sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n$$

is called the Taylor series of f around c. If $c=0$, then the Taylor series of f around c is called the **Maclaurin series** of f.

Let $R_n(x)=f(x)-P_n(x)$ for $x\in[a,b]$. If $R_n(x)\to 0$ for some $x\in[a,b]$, then the nth partial sum $P_n(x)$ of the Taylor series of f around c converges to $f(x)$, i.e., $f(x)$ is the sum of the Taylor series.

Now, the following two questions arise.

- At what points $x\in[a,b]$, the Taylor series of f around c converge?
- If the Taylor series of f converges for some x, does it converge to $f(x)$?

If $x=c$, then $P_n(c)=f(c)$ for all $n=0,1,\ldots$, and hence, $R_n(c)=0$. Therefore, the Taylor series of f converges to $f(c)$ at $x=c$. Consider the following examples.

(1) Let $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=|x|$. Then the Taylor series of f around 1 is $1+(x-1)$. But this is not equal to $f(x)$ when $x<0$.

(2) As an extreme case, consider $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x)=\begin{cases}e^{-1/x^2} & \text{if } x\neq 0,\\ 0 & \text{if } x=0.\end{cases}$$

Show that f is infinitely differentiable on $\mathbb{R}$ with $f^{(n)}(0)=0$ for all $n=0,1,\ldots$. Thus, the Maclaurin series of f is identically zero and it does not converge to $f(x)$ at $x\neq 0$.

Taylor's theorem helps us to show the convergence of a Taylor series of f to $f(x)$ in the following way. Recall that Taylor's theorem says that for $x\in[a,b]$ with $x\neq c$ and each $n\in\mathbb{N}$,

$$R_n(x)=\frac{f^{(n+1)}(c_{x,n})}{(n+1)!}(x-c)^{n+1}\text{ for some } c_{x,n} \text{ between } c \text{ and } x.$$

It is clear that if $R_n(x)\to 0$, then the Taylor series of f around c converges to $f(x)$.

Problem 6. *Show that the Maclaurin series of $f(x)=e^x$ converges to $f(x)$ for all $x\in\mathbb{R}$.*

Solution. By Taylor's theorem, there exists $c_{x,n}$ between 0 and x such that

$$e^x=1+x+\frac{x^2}{2!}+\cdots+\frac{e^{c_{n,x}}}{(n+1)!}x^{n+1}.$$

Since e^x is increasing, $e^{c_{n,x}}$ lies between $e^0=1$ and e^x. If $x<0$, then $c_{n,x}<0$, and $e^{c_{n,x}}<1$. If $x=0$, then $e^x=1$ and $R_n(x)=0$. If $x>0$, then $c_{n,x}>0$ and $e^{c_{n,x}}<e^x$.

Thus,

$$\left|\frac{e^{c_{n,x}}}{(n+1)!}x^{n+1}\right| \le \frac{|x|^{n+1}}{(n+1)!} \text{ if } x \le 0, \text{ and } \left|\frac{e^{c_{n,x}}}{(n+1)!}x^{n+1}\right| \le e^x \frac{x^{n+1}}{(n+1)!} \text{ if } x > 0.$$

Finally, $\lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0$ for every x. Hence, $\lim_{n\to\infty} R_n(x) = 0$. Therefore, the Maclaurin series of e^x converges to e^x for every x. Thus,

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

□

Exercise 7. *Show that for every $x \in \mathbb{R}$,*

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

and

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots .$$

References

[1] A. Baker, *Transcendental Number Theory*, Cambridge University Press, Cambridge, 1975.

[2] G. Chrystal, *Algebra: An Elementary Text-book for the Higher Classes of Secondary Schools and for Colleges*, sixth ed., parts I and II, Chelsea, New York, 1959.