CONVERGENCE TESTS FOR SERIES

In this lecture we will give several tests to determine the convergence of a series.

Theorem 1 (Comparison Test)**.** *Suppose $|a_n| \leq b_n$ for all $n \geq k$ for some k.*

(1) If $\sum b_n$ is convergent, then $\sum a_n$ is absolutely convergent and $|\sum a_n| \leq \sum b_n$.

(2) If $\sum |a_n|$ is divergent, then $\sum b_n$ is also divergent.

Examples.

(1) $\sum \frac{1}{n!}$ is convergent as $n^2 < n!$ for $n \geq 4$.

(2) $\sum \frac{1}{\sqrt{n}}$ diverges because $\frac{1}{n} \leq \frac{1}{\sqrt{n}}$.

(3) $\sum \frac{2^n+n}{3^n+n}$ converges as $\frac{2^n+n}{3^n+n} \leq \frac{2^n+2^n}{3^n} = 2(\frac{2}{3})^n$.

Theorem 2 (Limit Comparison Test)**.** *Suppose $a_n, b_n > 0$ eventually (i.e., $\exists\ k \in \mathbb{N}$ such that $a_n, b_n > 0\ \forall\ n \geq k$), and $\frac{a_n}{b_n} \to \ell$ as $n \to \infty$. Then*

(1) If $\ell > 0$, then $\sum a_n$ is convergent if and only if $\sum b_n$ is convergent.

(2) If $\ell = 0$ and $\sum b_n$ converges, then $\sum a_n$ also converges.

(3) If $\ell = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ also diverges.

Examples.

(1) $\sum \frac{2^n+n}{3^n-n}$ converges. If we take $b_n = (\frac{2}{3})^n$, then $\frac{a_n}{b_n} \to 1$.

(2) $\sum \sin(\frac{1}{n})$. Take $b_n = \frac{1}{n}$.

(3) $\sum \frac{1}{(\log n)^p}$ is divergent for $p > 0$. Let $b_n = \frac{1}{n}$. Then $\frac{a_n}{b_n} = \frac{1/(\log n)^p}{1/n} = \frac{n}{(\log n)^p} \to \infty$.

Theorem 3 (Cauchy condensation test)**.** *Let $a_n \geq 0$ and $a_{n+1} \leq a_n$. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges.*

Examples.

(1) $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$ (Verify).

(2) $\sum \frac{1}{n(\log n)^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Theorem 4 (Cauchy's Root Test)**.** *(1) If $|a_n|^{1/n} \leq \alpha$ eventually for some $\alpha < 1$, then $\sum a_n$ is absolutely convergent.*

(2) If $|a_n|^{1/n} \geq 1$ for infinitely many n, then $\sum a_n$ is divergent.

(3) In particular, if $|a_n|^{1/n} \to \ell$ where $\ell \in \mathbb{R}$ or $\ell = \infty$, then

$\sum a_n$ is absolutely convergent when $\ell < 1$, and it is divergent when $\ell > 1$.

Examples.

(1) $\sum \frac{1}{(\log n)^n}$ converges because $a_n^{1/n} = \frac{1}{\log n} \to 0$.
(2) $\sum (\frac{n}{n+1})^{n^2}$ converges as $a_n^{1/n} = \frac{1}{(1+\frac{1}{n})^n} \to \frac{1}{e} < 1$.

Theorem 5 (D'Alembert's Ratio Test). *Suppose $a_n \neq 0$ for all n.*

(1) If $|\frac{a_{n+1}}{a_n}| \leq \alpha$ eventually for some $\alpha < 1$, then $\sum a_n$ is absolutely convergent.
(2) If $|\frac{a_{n+1}}{a_n}| \geq 1$ eventually, then $\sum a_n$ is divergent.
(3) In particular, if $|\frac{a_{n+1}}{a_n}| \to \ell$ where $\ell \in \mathbb{R}$ or $\ell = \infty$, then
$\sum a_n$ is absolutely convergent when $\ell < 1$, and it is divergent when $\ell > 1$.

Remark 6. *Both Root test and Ratio test are inconclusive if $\ell = 1$.*

Examples.

(1) $\sum \frac{1}{n!}$ converges because $\frac{a_{n+1}}{a_n} \to 0$.
(2) $\sum (\frac{n^n}{n!})^{n^2}$ diverges as $\frac{a_{n+1}}{a_n} = (1 + \frac{1}{n})^n \to e > 1$.

Theorem 7 (Dirichlet's Test). *Let (a_n) and (b_n) be sequences such that (a_n) is monotonic, $a_n \to 0$, and the sequence (B_n) defined by $B_n = \sum_{k=1}^n b_k$ is bounded. Then the series $\sum a_n b_n$ is convergent.*

Corollary 8 (Leibniz Test). *Let (a_n) be a monotonic sequence such that $a_n \to 0$. Then $\sum (-1)^{n-1} a_n$ is convergent.*

Examples. For $p > 0$, both the series $\sum \frac{(-1)^{n-1}}{n^p}$ and $\sum \frac{(-1)^{n-1}}{(\log n)^p}$ are convergent.

Rearrangements of a Series

Let $\sum a_n$ and a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ be given. Define $b_n = a_{\sigma(n)}$. Then the new series $\sum b_n$ is said to be a rearrangement of the series $\sum a_n$.

Theorem 9 (Riemann's Theorem). *A conditionally convergent series can be made to converge to any arbitrary real number or even made to diverge by a suitable rearrangement of its terms.*

The above theorem should convince us the danger of manipulating a series without paying attention to rigorous analysis. The next theorem tells us when we can manipulate the terms of the series whichever way we want.

Theorem 10 (Rearrangement of Terms). *A series $\sum a_n$ is absolutely convergent if and only if every rearrangement of it is convergent. In this case, the sum of a rearrangement is unchanged.*