SERIES

We know that the set of real numbers is a group under the usual addition. The associativity of real numbers allows us to add any finite real numbers $x_1, x_2, \ldots, x_n$ as $x_1 + x_2 + \cdots + x_n$. In this lecture we will learn whether we can add infinitely many real numbers. In other words, if (x_n) is a sequence, then what is the meaning of the symbol $\sum_{n=1}^{\infty} x_n = x_1 + x_2 + \cdots + x_n + \cdots$. Such expression is called an infinite series, or just s series. What will happen if we try to add them term by term. For example, consider the sequence, $x_n = (-1)^{n-1}$. Then

$$\begin{aligned}
1 - 1 + 1 - 1 + \cdots &= (1-1) + (1-1) + \cdots \\
&= 0 + 0 + \cdots + 0 + \cdots \\
&= 0, \\
1 - 1 + 1 - 1 + \cdots &= 1 + (-1+1) + (-1+1) + \cdots \\
&= 1 + 0 + 0 + \cdots + 0 + \cdots \\
&= 1.
\end{aligned}$$

This absurdity shows that we should define the sum of infinite real numbers in a rigorous way so as to avoid this. In your school days, you have learned the geometric series $1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} + \cdots = 2$. What do we mean by this?

Convergence and Sum of an Infinite Series

Definition 1. *Let (x_n) be a sequence of real numbers. Define the sequence*

$$S_n = x_1 + x_2 + \cdots + x_n.$$

Then (S_n) is called the ***sequence of (n-th) partial sum*** *of the series $\sum_{n=1}^{\infty} x_n$.*

We say that the infinite series $\sum_{n=1}^{\infty} x_n$ is ***convergent*** *if the sequence (S_n) of partial sums is convergent.*

The limit of (S_n), say S, is called the sum of the series. We denote this fact by the symbol $\sum_{n=1}^{\infty} x_n = S$.

We say that the series $\sum_{n=1}^{\infty} x_n$ is ***divergent*** *if the sequence of its partial sums is divergent.*

The series $\sum_{n=1}^{\infty} x_n$ is said to be ***absolutely convergent*** *if the infinite series $\sum_{n=1}^{\infty} |x_n|$ is convergent.*

If a series is convergent but not absolutely convergent, then it is said to be **conditionally convergent***.*

You need to remember the following remark by heart.

Remark 2. *The ONLY way to deal with an infinite series is through its sequence partial sums and by using the definition of the sum of an infinite series.*

You need to be careful when dealing with infinite series. Mindless algebraic manipulations may lead to absurdities as shown in the beginning of this lecture.

Examples.

(1) **(Geometric Series)** Let $x \in \mathbb{R}$ such that $|x| < 1$. Consider the series $\sum_{n=0}^{\infty} x^n$. The sequence of partial sums is

$$S_n = 1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}.$$

Since $|x| < 1$, we have $x^{n+1} \to 0$, and therefore, $S_n \to \frac{1}{1-x}$. Thus, $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ if $|x| > 1$.

(2) The series $\sum_{n=0}^{\infty} \log(\frac{n+1}{n})$ diverges because $S_n = \log(n+1)$ diverges.

(3) **(Telescoping Series)** Let (x_n) and (y_n) be two sequences such that $x_n = y_{n+1} - y_n$. If S_n is the sequence of partial sums of $\sum_{n=0}^{\infty} x_n$, then

$$S_n = x_1 + x_2 + \cdots + x_n = (y_2 - y_1) + (y_3 - y_2) + \cdots + (y_{n+1} - y_n) = y_{n+1} - y_1.$$

This implies that $\sum_{n=0}^{\infty} x_n$ converges if and only if the sequence (y_n) converges. In this case, $\sum_{n=0}^{\infty} x_n = \lim_{n\to\infty} y_n - y_1$.

(4) Consider $\sum_{n=0}^{\infty} \frac{n}{n^4+n^2+1}$. Observe that $x_n = \frac{n}{n^4+n^2+1} = \frac{1}{2}\left[\frac{1}{n^2-n+1} - \frac{1}{n^2+n+1}\right] = \frac{1}{2}(y_n - y_{n+1})$, where $y_n = \frac{1}{2}\left(\frac{1}{n^2-n+1}\right)$. Hence, $S_n = \frac{1}{2} - \left(\frac{1}{n^2-n+1}\right) \to \frac{1}{2}$.

Necessary condition for convergence

Theorem 3 (The nth Term Test). *If $\sum_{n=0}^{\infty} x_n$ converges, then $x_n \to 0$.*

Proof. Let $\sum_{n=0}^{\infty} x_n = S$. Then $S_{n+1} - S_n = x_{n+1} \to S - S = 0$. □

Examples.

(1) If $|x| \geq 1$, the the geometric series $\sum_{n=0}^{\infty} x^n$ diverges because $x^n \nrightarrow 0$.

(2) $\sum_{n=0}^{\infty} \sin n$ diverges because $\sin n \nrightarrow 0$.

(3) $\sum_{n=0}^{\infty} \log(\frac{n+1}{n})$ diverges, however, $\log(\frac{n+1}{n}) \to 0$.

Necessary and sufficient condition for convergence

Theorem 4. *Suppose $x_n \geq 0$ for all n. Then $\sum_{n=0}^{\infty} x_n$ converges iff (S_n) is bounded above.*

Proof. Exercise. □

Example. The Harmonic series $\sum_{n=0}^{\infty} \frac{1}{n}$ diverges. To see this, we will show that the sequence of partial sums (S_n) is not bounded above. It is enough to show that the subsequence S_{2^k} is not bounded above (Why?). Observe that

$$\begin{aligned} S_1 &= 1 \\ S_2 &= 1 + \frac{1}{2} \\ S_{2^2} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) \\ &> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) \\ &= 1 + \frac{1}{2} + 2 \cdot \frac{1}{4} = 1 + 2 \cdot \frac{1}{2} \\ S_{2^3} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \\ &> 1 + \frac{1}{2} + 2 \cdot \frac{1}{4} + 2^2 \cdot \frac{1}{2^3} = 1 + 3 \cdot \frac{1}{2}. \end{aligned}$$

This implies that

$$S_{2^k} > 1 + k \cdot \frac{1}{2}.$$

Thus, (S_n) is not bounded above, and hence the series $\sum_{n=0}^{\infty} \frac{1}{n}$ diverges.

Let's see another proof. Assume that (S_n) is convergent. Then (S_n) is Cauchy. It follows that for $\varepsilon = \frac{1}{2}$, there exists $N \in \mathbb{N}$ such that $|S_{2m} - S_m| < \frac{1}{2}$ for all $m \geq N$. But

$$|S_{2m} - S_m| = \frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{2m} > \frac{1}{2m} + \frac{1}{2m} + \cdots + \frac{1}{2m} > \frac{1}{2}.$$

This is a contradiction. Hence, (S_n) cannot converge.

Algebra of Convergent Series

Given two series (whether or not convergent) $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$, and a scalar $\alpha \in \mathbb{R}$, we define

$$\sum_{n=1}^{\infty} x_n + \sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} (x_n + y_n) \text{ and } \alpha \sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} (\alpha x_n).$$

The following theorem shows that the set of all convergent series is a vector space over $\mathbb{R}$. The proof is straightforward and you should go for it.

Theorem 5. *Let $\sum_{n=1}^{\infty} x_n = x$ and $\sum_{n=1}^{\infty} y_n = y$. Then*

(1) *Their sum* $\sum_{n=1}^{\infty}(x_n + y_n)$ *is convergent and* $\sum_{n=1}^{\infty}(x_n + y_n) = x + y$.
(2) *The series* $\alpha \sum_{n=1}^{\infty} x_n$ *is convergent and* $\alpha \sum_{n=1}^{\infty} x_n = \alpha x$.

We now give an important result about absolutely convergent series.

Theorem 6. *An absolutely convergent series is convergent.*

Proof. Let S_n and Γ_n denote the partial sums of $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} |x_n|$ respectively. For $n > m$, we have

$$|S_n - S_m| = |\sum_{k=m+1}^{n} x_k| \leq \sum_{k=m+1}^{n} |x_k| = \Gamma_n - \Gamma_m,$$

which converge to 0 as (Γ_n) is convergent. Therefore, (S_n) is Cauchy. □

We will see later that the converse of the above result does not hold.