TAYLOR'S THEOREM

Taylor Polynomials

Sometimes we can approximate complicated functions with simpler ones that give the accuracy we want for specific applications and are easier to work with. The approximating functions discussed here are called linearizations, and they are based on tangent lines. Other approximating functions, such as polynomials will be discussed afterwards.

Consider the function $f(x) = x^2$. The tangent to the curve $y = x^2$ at $(1, 1)$ is $y = 2x-1$. If we plot the graph of the curve and its tangent, we can see that the tangent line lies close to the curve near the point of tangency. If we zoom the graph of f near $(1, 1)$, the graph becomes flatter and it almost resembles its tangent. Thus, for a small interval around the point 1 on x-axis, the y-values along the tangent line gives a good approximations to the y-values on the curve.

In general, the tangent to $y = f(x)$ at a point c, where f is differentiable, is the line

$$L(x) = f(c) + f'(c)(x - c).$$

As long as this line remains close to the graph of f, $L(x)$ provides a good approximation to $f(x)$. The approximating function $L(x)$ is called the linearization of f at c. This is the standard linear approximation of f at c, and the point c is the center of the approximation.

Note that the linearization $L(x)$ of f at c is a polynomial of degree one. The question here is that can we get a better approximation if we take a polynomial of higher degree. The answer is yes and this is what Taylor's theorem talks about[1].

Definition 1. *Let $f : [a, b] \to \mathbb{R}$ be a function which is n-times differentiable at $c \in (a, b)$. The Taylor polynomial of order n generated by f at c is the polynomial*

$$P_n(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n.$$

Example 2. *Find the Taylor polynomial generated by $f(x) = e^x$ at $c = 0$.*

Solution. Since $f^{(n)}(x) = e^x$ and $f^{(n)}(0) = 1$ for every n, the Taylor polynomial of order n at $c = 0$ is

$$P_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}.$$

[1] Taylor's theorem is named after the English mathematician Brook Taylor. He also introduced Taylor series which will be discussed later. Both Taylor's theorem and Taylor series are among the most useful results in calculus.

Taylor's Theorem

Theorem 3. *Assume that $f : [a,b] \to \mathbb{R}$ is such that $f^{(n)}$ is continuous on $[a,b]$ and $f^{(n+1)}(x)$ exists on (a,b). Fix $x_0 \in [a,b]$. Then for each $x \in [a,b]$ with $x \neq x_0$, there exists c between x and x_0 such that*

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}. \qquad (\star)$$

Remark 4.

1. *Note that the MVT corresponds to the case $n = 0$ of Taylor's Theorem. The case $n = 1$ is sometimes called the Extended Mean Value Theorem. In both cases we take $x_0 = a$ and $x = b$.*
2. *The right-hand side of Equation $(\star)$ is called the n-th order Taylor expansion (or formula) for f around x_0.*
3. *The term $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$ is called the remainder term of order n. It is known as Lagrange's form of remainder in Taylor's formula.*
4. *Usually, the n-th order Taylor polynomial $P_n(x)$ of f around x_0 provides a progressively better approximation to f around x_0 as n increases. The remainder term is the "error term" if we wish to approximate f near x_0 by $P_n(x)$. If we assume that $f^{(n+1)}$ is bounded by M on (a,b), then R_n goes to 0 much faster than $(x - x_0)^n \to 0$, since $\left|\frac{R_n(x)}{(x-x_0)^n}\right| \leq \frac{M}{(n+1)!}|x - x_0|$.*

Corollary 5. *Let $f : [a,b] \to \mathbb{R}$ and n be a nonnegative integer. Then f is a polynomial function of degree $\leq n$ iff $f^{(n+1)}$ exists and is identically zero on $[a,b]$.*

Proof. Exercise. □

Example 6. *Using Taylor's theorem show that for any $k \in \mathbb{N}$ and for all $x > 0$,*

$$x - \frac{1}{2}x^2 + \cdots + \frac{1}{2k}x^{2k} < \log(1+x) < x - \frac{1}{2}x^2 + \cdots + \frac{1}{2k+1}x^{2k+1}.$$

Solution. By Taylor's theorem, there exists $c \in (0,x)$ such that

$$\log(1+x) = x - \frac{1}{2}x^2 + \cdots + \frac{(-1)^{n-1}}{n}x^n + \frac{(-1)^n}{n+1}\frac{1}{(1+c)^{n+1}}x^n.$$

We observe that for any $x > 0$,

$$\frac{(-1)^n}{n+1}\frac{1}{(1+c)^{n+1}}x^n = \begin{cases} > 0 & \text{if } n = 2k \\ < 0 & \text{if } n = 2k+1. \end{cases}$$

Problem 7. *Let $x_0 \in (a,b)$ and $n \geq 2$. Suppose $f', f", \ldots, f^{(n)}$ are continuous on (a,b) and $f'(x_0) = \cdots = f^{(n-1)}(x_0) = 0$. Show that if n is even and $f^{(n)}(x_0) > 0$, then f has a local minimum at x_0. Similarly, if n is even and $f^{(n)}(x_0) < 0$, show that f has a local maximum at x_0.*

Solution. Suppose that $f^{(n)}(x_0) > 0$ and n is even. Since $f^{(n)}$ is continuous at x_0, there exists a neighbourhood U of x_0 such that $f^{(n)}(x) > 0$ for all $x \in U$. By Taylor's theorem, for $x \in U$, there exists c between x and x_0 such that $f(x) = f(x_0) + \frac{f^{(n)}(c)}{n!}(x - x_0)^n$. It follows that $\frac{f^{(n)}(c)}{n!}(x - x_0)^n > 0$ ($\because$ $c \in U$ and n is even). This implies that $f(x) > f(x_0)$ for all $x \in U$. Hence, x_0 is a local minimum.