THE PICARD AND NEWTON METHODS

In this lecture we will study a numerical method which is a technique to find the approximate solution of the equation $f(x) = 0$. This is the next best alternative to finding an exact solution. In fact, finding an exact solution is a very difficult problem even for the nicest functions, namely polynomials[1].

Consider the following two problems.

(1) Let $f : [a, b] \to [a, b]$ be a function. Does there exist a point $x \in [a, b]$ such that $f(x) = x$. Such a point is called a fixed point of f. Fixed point theory is one of the most powerful tools of modern mathematics.

(2) Let $f : [a, b] \to \mathbb{R}$ be a function. Does there exist a point $x \in [a, b]$ such that $f(x) = 0$.

We first observe that these two problems are interrelated in the sense that if you solve one, you solve the other. If $f : [a, b] \to [a, b]$, and if we choose $F(x) = f(x) - x$, then finding a fixed point of f is equivalent to finding a solution to $F(x) = 0$. On the other hand, if $f : [a, b] \to \mathbb{R}$ and we set $F(x) = x + h(x)g(x)$ where $h : [a, b] \to \mathbb{R}$ is chosen such that $h(x) \neq 0$ and $x + h(x)g(x) \in [a, b]$ for all $x \in [a, b]$, then finding a solution of $f(x) = 0$ is equivalent to finding a fixed point of $F : [a, b] \to [a, b]$.

Proposition 1. *If $f : [a, b] \to [a, b]$ is continuous, then f has a fixed point.*

Proof. Define $F : [a, b] \to \mathbb{R}$ by $F(x) = f(x) - x$. Now,

$$F(a) = f(a) - a \geq 0 \text{ and } F(b) = f(b) - b \leq 0.$$

By IVP, there exists $c \in [a, b]$ such that $F(c) = 0$, i.e., $f(c) = c$. □

1. Picard method

Suppose that a function $f : [a, b] \to [a, b]$ has a fixed point. The question is that can we find it. In general this is so easy. So, we try to find it approximately. A simple and effective method is given by Picard which is described in the algorithm below.

[1]For linear and quadratic equations, there are simple and well-known formulas for their solutions. For cubic and quadratic equations, there are complicated formulas due to Cardan and Ferrari, which express the solutions in terms of the coefficients of the polynomial. For a general polynomial equation of degree 5 or more, Abel proved that no such formula exists. Galois' theory provides a much more complete answer to this question, by explaining why it is possible to solve some equations, and why it is not possible for most equations of degree 5 or higher.

Algorithm 1. Given any $x_0 \in [a, b]$, define (x_n) by

$$x_n = f(x_{n-1}) \text{ for } n \in \mathbb{N}.$$

Such a sequence is called Picard sequence for the function f (with its initial point x_0). It is clear that if (x_n) is convergent and f is continuous, then the limit x of (x_n) is a fixed point of f. Indeed,

$$f(x) = f(\lim_{n\to\infty} x_n) = f(\lim_{n\to\infty} x_{n-1}) = \lim_{n\to\infty} f(x_{n-1}) = \lim_{n\to\infty} x_n = x.$$

The next theorem gives a sufficient condition for the convergence of a Picard sequence.

Theorem 2 (Picard Convergence Theorem)**.** *If $f : [a, b] \to [a, b]$ is continuous on $[a, b]$ and differentiable on (a, b) with $|f'(x)| < 1$ for all $x \in (a, b)$, then f has a unique fixed point. Furthermore, any Picard sequence for f is convergent and converges to the unique fixed point of f.*

Example. Let $f : [0, 2] \to \mathbb{R}$ be defined as $f(x) = (1 + x)^{1/5}$. Then $0 \le f(x) \le 3^{1/5} < 2$ for all $x \in [0, 2]$. Thus, f maps the interval $[0, 2]$ into itself. Moreover, $|f'(x)| = \frac{1}{5(1+x)^{4/5}} \le \frac{1}{5} < 1$ for $x \in [0, 2]$. Picard convergence theorem implies that f has a unique fixed point. In other words, the sequence (x_n) defined by $x_{n+1} = (1 + x_n)^{1/5}$ converges to a root of $x^5 - x - 1 = 0$ in the interval $[0, 2]$.

2. Newton–Raphson method

Suppose we know that $f : [a, b] \to \mathbb{R}$ is such that the equation $f(x) = 0$ has a solution. We have discussed earlier that it is difficult to find an exact solution. So, we find an approximate solution. A method given by Newton is used to achieve this.

Algorithm 2. Choose any $x_0 \in [a, b]$ such that $f'(x_0)$ exists and $f'(x_0) \neq 0$. Given any $n \in \mathbb{N}$ and $x_{n-1} \in [a, b]$ such that $f'(x_{n-1}) \neq 0$, let

$$x_n = x_{n-1} - \frac{f(x_n - 1)}{f'(x_{n-1})}.$$

Such a sequence (x_n) is called a Newton sequence for the function f (with its initial point x_0). It is easy to verify that if a Newton sequence (x_n) for f is convergent and f' is bounded, then the limit x of (x_n) satisfies $f(x) = 0$. We can see that if we choose $f(x) = x - \frac{f(x)}{f'(x)}$, then algorithm 2 is a particular case of algorithm 1.

Example. Suppose $f(x) = x^2 - a$, $a > 0$. Since $f'(x) = 2x$, the Newton sequence for the function f is $x_{n+1} = \frac{1}{2}(x_n + \frac{a}{x_n})$. We have seen in Lecture 2 that this sequence converges to $\sqrt{a}$.

We end this lecture by giving a sufficient condition for the convergence of a Newton sequence.

Proposition 3. *Let $f : [a,b] \to \mathbb{R}$ be such that $f(c) = 0$ for some $[a,b]$. If f' is nonzero and monotonic throughout $[a,b]$, then c is the unique solution of $f(x) = 0$ in $[a,b]$ and the Newton sequence with any initial point $x_0 \in [a,b]$ converges to c.*