LOCAL EXTREMA AND POINTS OF INFLECTION

In lecture 8, we have seen a necessary condition for local maximum and local minimum. In this lecture we will see some sufficient conditions.

In the following results we assume $f : (a, b) \to \mathbb{R}$ and $c \in (a, b)$.

1. Sufficient conditions for a local extremum

We will state results for local maximum, and results for local minimum are similar.

Theorem 1. *Let f be continuous at c. If for some $\delta > 0$, f is increasing on $(c - \delta, c)$ and decreasing on $(c, c + \delta)$, then f has a local maximum at c.*

Proof. Choose x and y such that $c - \delta < x < y < c$. Then $f(x) \leq f(y)$. The continuity of f at c implies that $f(x) \leq \lim_{y \to c^-} f(y) = f(c)$. Similarly, if $c < y < x < c + \delta$, then $f(x) \leq \lim_{y \to c^+} f(y) = f(c)$. This proves the result. □

Corollary 2. (1) ***(First Derivative Test for Local Maximum)*** *Let f be continuous at c. If for some $\delta > 0$*

$$f'(x) \geq 0 \ \forall \ x \in (c - \delta, c) \text{ and } f'(x) \leq 0 \ \forall \ x \in (c, c + \delta),$$

then f has a local maximum at c.

(2) ***(Second Derivative Test for Local Maximum)*** *If f is twice differentiable at c and satisfies $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.*

Remark 3. *An easy way to remember the First Derivative Test (for local minimum and local maximum) is as follows:*

f' changes from $-$ to $+$ at $c \Rightarrow f$ has a local minimum at c,

f' changes from $+$ to $-$ at $c \Rightarrow f$ has a local maximum at c.

Examples

(1) Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = \frac{1}{x^4 - 2x^2 + 7}$. We have $f'(x) = \frac{-4x(x-1)(x+1)}{(x^4 - 2x^2 + 7)^2}$. Thus, $f'(x) = 0$ when $x = -1, 0, 1$. Now, consider the following table,

Interval	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
Sign of f'	$+$	$-$	$+$	$-$

So, we conclude that f has a local minimum at $x = 0$ and a local maxima at $x = -1$ and $x = 1$.

(2) Consider $f : (-1,1) \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 & \text{if } 0 < |x| < 1, \\ -1, & \text{if } x = 0. \end{cases}$$

We note the conditions of the first derivative test is not satisfied. In fact, f is differentiable on $(-1,0)$ and $(0,1)$ and f' changes sign from $-$ to $+$ at $x = 0$ but f is not continuous at $x = 0$. Nevertheless, $f(0) < f(x)$ for all nonzero $x \in (-1,1)$, and thus f has a strict local minimum at $x = 0$.

(3) If $f : \mathbb{R} \to \mathbb{R}$ is defined as $f(x) = x^4$. Then $f(0) = 0 < f(x)$ for all nonzero $x \in \mathbb{R}$). Therefore, f has a strict local minimum at $x = 0$. Note that $f'(0) = 0$, but $f''(0)$ is not positive.

2. Convex Sets and Convex Functions

Let V be a vector space over $\mathbb{R}$.

Definition 4. *A set $C \subseteq V$ is said to be convex if the line segment between any two points in C lies in C, i.e.,*

if for any $x, y \in C$ and any $t \in [0,1]$, we have $tx + (1-t)y \in C$.

Example. It is clear that the unit disc is convex in $\mathbb{R}^2$. However, the unit circle is not convex. Any interval in $\mathbb{R}$ is a convex set.

Definition 5. *Let $C \subseteq V$ be a convex set. A function $f : C \to \mathbb{R}$ is said to be **convex** if for all $x, y \in C$ and for all $t \in [0,1]$, we have*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y). \tag{$\star$}$$

If for $t \in (0,1)$, the above inequality is strict, the f is said strictly convex.

*We say that f is **concave** if the reverse inequality in $(\star)$ holds.*

Theorem 6 (Derivative Test for Convexity)**.** *Assume that $f : [a,b]$ is differentiable on (a,b). If f' is increasing on (a,b), then f is convex on $[a,b]$. In particular, if f'' exists and non-negative on (a,b), then f is convex.*

Example. Let $f(x) = x^3 - 6x^2 + 9x$. We have $f'(x) = 3(x-1)(x-3)$ and $f''(x) = 6x - 12$. We see that $f''(x) > 0$ if $x > 2$ and $f''(x) < 0$ if $x < 2$. Hence, f is convex for $x > 2$ and concave for $x < 2$.

Examples of convex functions

- e^x is strictly convex on $\mathbb{R}$.
- $x \log x$ is strictly convex on $(0, \infty)$.

- $f(x) = x^4$ is strictly convex but $f''(0) = 0$

The following result is one of the reasons why convex functions are very useful in applications especially in optimization problems.

Theorem 7. *If $f : (a, b) \to$ is convex and $c \in (a, b)$ is a local minimum, then c is a minimum for f on (a, b). That is, local minima of convex functions are global minima.*

3. Points of Inflection

Definition 8. *Let $f : (a, b) \to \mathbb{R}$ be a function and $c \in (a, b)$. The point c is said to be a* ***point of inflection*** *for f if there is $\delta > 0$ such that f is convex in $(c - \delta, c)$, while f is concave in $(c, c + \delta)$, or vice versa, that is, f is concave in $(c - \delta, c)$, while f is convex in $(c, c + \delta)$.*

Examples For the function $f(x) = x^3$ on $\mathbb{R}$, 0 is a point of inflection.

Theorem 9. *Let $f : (a, b) \to \mathbb{R}$ be a function and $c \in (a, b)$.*

(1) ***[Necessary Condition for a Point of Inflection]*** *Let f be twice differentiable at c. If c is a point of inflection for f, then $f''(c) = 0$.*
(2) ***[Sufficient Condition for a Point of Inflection]*** *Let f be thrice differentiable at c. If $f''(c) = 0$ and $f'''(c) \neq 0$, then c is a point of inflection for f.*

Examples

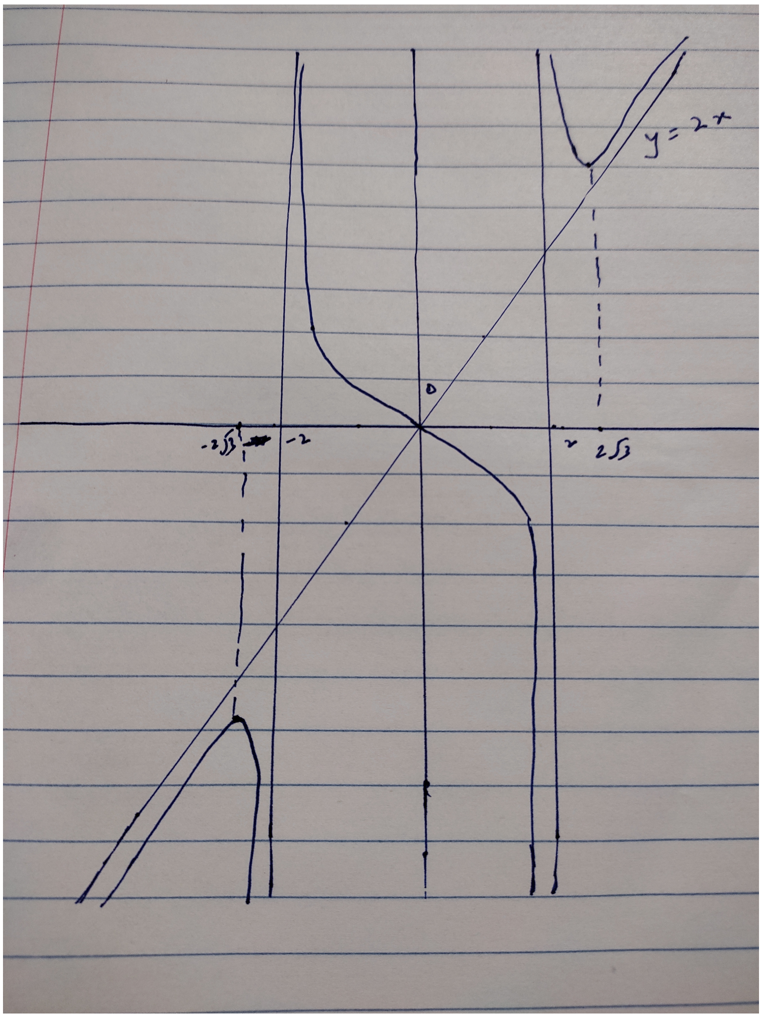
- For the function $f(x) = x^4$, 0 is not a point of inflection, though, $f''(0) = 0$.
- For the function $f(x) = x^5$, 0 is a point of inflection, but $f'''(0) = 0$.

Problem 10. *Sketch the graph of the function $f(x) = \frac{2x^3}{x^2-4}$ after finding the intervals of decrease/increase, intervals of concavity/convexity, points of local minima/local maxima, points of inflection and asymptotes.*

Solution. We note that

$$f(x) = 2x + \frac{8x}{x^2 - 4}, \quad f'(x) = \frac{2x^2(x^2 - 12)}{(x^2 - 4)^2} \text{ and } f''(x) = \frac{16x(x^2 + 12)}{(x^2 - 4)^3}.$$

Verify that $x = 2$, $x = -2$ and $y = 2x$ are the asymptotes. Moreover, the function is increasing on $(-\infty, -2\sqrt{3})$ and $(2\sqrt{3}, \infty)$. The function is decreasing on $(-2\sqrt{3}, -2)$, $(-2, 2)$ and $(2, 2\sqrt{3})$. Furthermore, the function is convex on $(-2, 0)$ and $(2, \infty)$ and concave on $(-\infty, -2)$ and $(0, 2)$. The point of inflection is 0. The sketch of the graph is shown below.


y = 2x
0
-2√3
-2
2
2√3

□