QUANTIFIERS

In this lecture, we will learn about common sense logic and the use of quantifiers in mathematical statements. We have been using statements that involve the quantifier $\forall$ (called the universal quantifier), $\exists$ (called the existential quantifier) and logical connectives *and/or* since our school days. In the course of Linear Algebra we have seen how we prove a statement by contradiction and how to establish a result in contrapositive form.

Let us look at the following sentences. All the notations are standard and same as used in lectures.

- There exists a proper subspace of $\mathbb{R}$.
- There is a vector space which does not have any basis.

We know that both of the statements above are false. How do we prove that? What is the negation of these statements? If we want to prove that the first statement is false, we have to show that any proper subset of $\mathbb{R}$ is not a subspace. For the second one, we have to show that any vector space has a basis. Look at the difference carefully.

The above two statements can be abstracted as follows. There exists an element in a set X which has some property P. In terms of quantifiers, we write

$$\text{“}\exists\ x \in X\ (x \text{ has property } P)\text{”}$$

To prove that this sentence is false means that we negate the sentence. The negation of the above statement is

$$\text{“}\forall\ x \in X\ (x \text{ does not have property } P)\text{”}$$

Let us come back to the two statements mentioned in the beginning of this note.

- There exists a proper subspace of $\mathbb{R}$. In terms of quantifiers
  “$\exists\ A \subset \mathbb{R}$ (A is a subspace)”
  The negation of the above statement is
  “$\forall\ A \subset \mathbb{R}$ (A is not a subspace)”
- There is a vector space which does not have any basis. In terms of quantifiers, we have the following: The negation of
  “$\exists$ a vector space V (V does not have a basis)”
  is “$\forall$ vector spaces V (V has a basis)”

The following remark is important.

Remark 1. *Remember that when you have to prove statements like “$\forall\ A \subset \mathbb{R}$ (A is not a subspace)” or “$\forall$ vector space V (V has a basis)”, you cannot do it by taking* *a particular example. In the former case, you have take an arbitrary subset of* $\mathbb{R}$ *and prove show that it's not a subspace. In the later case, you have to take an arbitrary vector space (finite or infinite dimensional!)* V *and prove that it has a basis.*

The above sentences use the existential quantifier $\exists$. Now let us look at sentences which use the universal quantifier $\forall$:

- Every 2×2 real matrix has zero determinant.
- Every subset of $\mathbb{R}^2$ is linearly independent.

In abstract form, the above sentences follow the pattern:

$$\text{“}\forall\ x \in X\ (x \text{ has property } P)\text{”}$$

The negation of this sentence is

$$\text{“}\exists\ x \in X\ (x \text{ does not have property } P)\text{”}$$

In terms of quantifiers, the above two statements can be written as

- "$\forall\ A \in M_2(\mathbb{R})\ (|A| = 0)$". Its negation is
  "$\exists\ A \in M_2(\mathbb{R})\ (|A| \neq 0)$"
- "$\forall\ A \subset \mathbb{R}^2$ (A is L.I.)". Its negation is
  "$\exists\ A \subset \mathbb{R}^2$ (A is L.D.)"

Remark 2. *If you have to prove the statements "*$\exists\ A \in M_2(\mathbb{R})\ (|A| \neq 0)$*" or "*$\exists\ A \subset \mathbb{R}^2$ *(*A *is L.D.)", what you have to do is to give an example of a* 2×2 *matrix whose determinant is not zero in the first one, and in the second one, you should give an example of a subset of* $\mathbb{R}^2$ *which is L.D.*

We now consider sentences which are combined using the connectives *and* or *or*.

Let G be a nonempty set with a binary operation $*$. Consider the following sentences.

- (Existence of identity) There exists an element $e \in G$ such that $a * e = e * a = a$ for all $a \in G$. In terms of quantifiers we write
  "$\exists\ e \in G\ (\forall\ a \in G\ (a * e = e * a = a)))$". Note that if we want to claim that this nested sentence is false we have to negate it. Its negation reads as
  "$\forall\ e \in G\ (\exists\ a \in G\ (a * e \neq e * a$ or $a * e \neq a$ or $e * a \neq a)))$".
- (Existence of inverse) For each element $a \in G$, there exists $b \in G$ such that $a * b = b * a = e$, where the e is the identity element. In terms of quantifiers we have
  "$\forall\ a \in G\ (\exists\ b \in G\ (a * b = b * a = e)))$". Its negation is
  "$\exists\ a \in G\ (\forall\ a \in G\ (a * b \neq b * a$ or $a * b \neq e$ or $b * a \neq e)))$".

Before proceeding further we just add a remark regarding the previous two examples. Let $x, y, z \in X$, where X in any set. Consider the statement

"$x = y = z$".

This is actually a combination of three statements,

$x = y$ and $y = z$ and $x = z$.

When we say that these combined statements are not true, it means that atleast one of them should be false. That is, $x \neq y$ or $y \neq z$ or $x \neq z$. Thus we arrive at the following: If a statement is of the form P and Q and R, then its negation is P is false or Q is false or R is false, that is, Not P or Not Q or Not R.

Exercise 3. *Let A, B, C be subsets of a set X. Try to write the negation of the following sentences.*

(1) $x \in A \cap B$.

(2) $x \in A \cup B$.

(3) $A \neq B$.

Exercise 4. *Negate the following sentence using quantifiers:*

"every road in the city has at least one home in which we can find a man who is wealthy and handsome or educated and generous."

Let us now deal with a more complicated sentence. Suppose there is a garden which is full of trees. We make the following statement.

"In each tree in the garden, there is a branch on which all of the leaves are red.

Try to negate this sentence using your common sense. You'll be in lot of trouble.

Let us turn this statement into a mathematical sentence? We first fix some notations. Let T denotes the set of all trees in the garden. For $t \in T$, let B_t denotes the set of all branches on the tree t. For $b \in B_t$, we let L_b denotes the set of all leaves on the branch b. Using quantifiers, the above sentence reads as

$$\text{“}\forall\ t \in T\ (\exists\ b \in B_t\ (\forall\ l \in L_b\ (l \text{ is red})))\text{”}$$

How do we negate it? We do it layer by layer. So,

$$\text{“}\exists\ t \in T\ (\forall\ b \in B_t\ (\exists\ l \in L_b\ (l \text{ is not red})))\text{”}$$

Now the question is that do such complicated sentence appear in Mathematics. Yes, when will study convergence of a sequence of real numbers.

Exercise 5. *Let A, B, C be three subsets of $\mathbb{R}$. What is the negation of the following statement:*

For every $\varepsilon > 1$, there exists $a \in A$ and $b \in B$ such that for all $c \in C$, $|a - c| < \varepsilon$ and $|b - c| > \varepsilon$.