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REFLEXIVITY OF ISOMETRIES OF ORDER n

BY

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Dedicated to the memory of Professor Sudipta Dutta

Abstract. We prove that if the group of isometries on $C_0(\Omega, X)$ is algebraically reflexive, then the set of isometries of order n on $C_0(\Omega, X)$ is also algebraically reflexive. Here, Ω is a first countable locally compact Hausdorff space, and X is a Banach space having the strong Banach–Stone property. As a corollary, we establish the algebraic reflexivity of the set of generalized bi-circular projections on $C_0(\Omega, X)$.

1. Introduction. Reflexivity and hyperreflexivity explore the relation between sets of operators and their common invariant subspaces. The first results on reflexivity were proved by D. Sarason [21]. The notion of reflexivity was introduced by Halmos [8] for lattices of closed subspaces of a Hilbert space $\mathcal{H}$. If $\mathcal{A}$ is a subset of $B(\mathcal{H})$, then $\operatorname{Lat}\mathcal{A}$ denotes the set of all subspaces invariant under every operator in $\mathcal{A}$. If $\mathcal{L}$ is a collection of closed subspaces of $\mathcal{H}$, then $\operatorname{Alg}\mathcal{L}$ denotes the algebra of all operators which leave every subspace in $\mathcal{L}$ invariant. A lattice $\mathcal{L}$ is *reflexive* if $\mathcal{L} = \operatorname{Lat}\operatorname{Alg}\mathcal{L}$. Reflexivity for algebras was introduced by Radjavi and Rosenthal [18]. An algebra $\mathcal{A}$ is called *reflexive* if $\mathcal{A} = \operatorname{Alg}\operatorname{Lat}\mathcal{A}$.

Loginov and Šul′man [15] extended this notion to linear subspaces of $B(\mathcal{H})$. For any subspace $\mathcal{S}$ of $B(\mathcal{H})$, define

$$\operatorname{Ref}\mathcal{S} = \{T \in B(\mathcal{H}) : Th \in \overline{\mathcal{S}h},\ \forall\, h \in \mathcal{H}\}. \tag{1.1}$$

$\operatorname{Ref}\mathcal{S}$ is called the *attached space* for $\mathcal{S}$ or the *topological closure* of $\mathcal{S}$. The subspace $\mathcal{S}$ is called *reflexive* (or *topologically reflexive*) if $\mathcal{S} = \operatorname{Ref}\mathcal{S}$.

Throughout this paper, let X be a Banach space and $B(X)$ the algebra of all bounded linear operators on X. One can see that in (1.1), the assumptions that the underlying space is a Hilbert space and $\mathcal{S}$ is a linear subspace are not essential. We can define the topological closure for an arbitrary subset $\mathcal{S}$ of $B(X)$.

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If $\mathcal{A}$ is an algebra that contains the identity, then $\operatorname{Lat}\mathcal{A}$ is determined by the closed cyclic subspaces of $\mathcal{A}$, so $\operatorname{Alg}\operatorname{Lat}\mathcal{A} = \operatorname{Ref}\mathcal{A}$. Thus, the two definitions coincide for unital algebras.

The notion of algebraic reflexivity has appeared in various contexts. The term was coined by Hadwin [7]. Let V be a vector space over a field $\mathbb{F}$, and let $\mathcal{L}(V)$ denote the algebra of all linear transformations on V. For a subspace $\mathcal{S}$ of $\mathcal{L}(V)$ define

$$\overline{\mathcal{S}}^a = \{T \in \mathcal{L}(V) : Tx \in \mathcal{S}x, \forall\, x \in V\}.$$

So, $T \in \overline{\mathcal{S}}^a$ if and only if for each $x \in V$, there exists $S \in \mathcal{S}$, depending on x, such that $Tx = Sx$. We say that T *interpolates* $\mathcal{S}$ or T is *locally in* $\mathcal{S}$. Obviously, $\mathcal{S} \subseteq \overline{\mathcal{S}}^a$. The subspace $\mathcal{S}$ is called *algebraically reflexive* if $\mathcal{S} = \overline{\mathcal{S}}^a$.

Obviously, any topologically reflexive subspace is algebraically reflexive.

Algebraic reflexivity in general and on certain classes of transformations were studied by many authors: see for instance [4, 5, 7, 12, 14, 17, 19, 20]. The lecture notes by Molnár [16] give a very comprehensive account of this theory.

An important class of transformations in $B(X)$ is the group of surjective linear isometries, denoted by $\mathcal{G}(X)$. We denote by $\mathcal{G}^n(X)$ the set of operators T in $\mathcal{G}(X)$ such that $T^n = I$, called *isometries of order* n. An operator $T \in \overline{\mathcal{G}(X)}^a$ is called a *local isometry*.

The isometry group of any finite-dimensional Banach space is algebraically reflexive. Every Banach space admits a renorming whose isometry group is algebraically reflexive [11]. The isometry group of any infinite-dimensional Hilbert space fails to be algebraically reflexive. Indeed, given $x, y \in \mathcal{H}$ such that $\|x\| = \|y\|$, there exists $T \in \mathcal{G}(\mathcal{H})$ such that $T(x) = y$. So, $\overline{\mathcal{G}(\mathcal{H})}^a$ contains all into isometries.

For a somewhat nontrivial but easy example, one can look at ℓ_∞ in $B(\ell_2)$. We know that $B(\ell_2)$ contains an isometric copy of ℓ_∞. We shall see later that ℓ_∞ is algebraically reflexive.

In [5], Dutta and Rao proved that for a compact Hausdorff space Ω, if $\mathcal{G}(C(\Omega))$ is algebraically reflexive, then so is $\mathcal{G}^2(C(\Omega))$. Motivated by this result, in this paper we investigate the algebraic reflexivity of isometries of order n on $C_0(\Omega, X)$, the space of X-valued continuous functions on a first countable locally compact Hausdorff space Ω vanishing at infinity.

A projection $P \in B(X)$ is said to be a *generalized bi-circular projection* if $P + \lambda(I - P) \in \mathcal{G}(X)$, where λ is a unit modulus complex number not equal to 1. This class was introduced by Fošner, Ilišević and Li [6] in 2007. Descriptions of generalized bi-circular projections for various Banach spaces can be found in [1, 3, 10, 13]. As a corollary to our result, we establish the algebraic reflexivity of the set of generalized bi-circular projections on $C_0(\Omega, X)$, which answers a question raised by Dutta and Rao [5].

2. Preliminaries. The study of isometries between Banach spaces is one of the most important research areas in functional analysis. One of the most classical results in this area is the Banach–Stone theorem describing surjective linear isometries between Banach spaces of complex-valued continuous functions on compact Hausdorff spaces.

While investigating reflexivity problems for the isometry group of a Banach space X, firstly we observe that any local isometry on X is actually an isometry. In particular, if $T \in \overline{\mathcal{G}(X)}^{a}$, then for any $x \in X$, there exists $T_x \in \mathcal{G}(X)$ such that $T(x) = T_x(x)$. Hence $\|T(x)\| = \|T_x(x)\| = \|x\|$. So, in order to show that $\mathcal{G}(X)$ is algebraically reflexive, we need to prove that any local isometry is surjective. But this is not as easy as it seems. Secondly, since we have a precise description of surjective isometries for most of the classical Banach spaces, one has a good idea of what any local isometry looks like.

For the sake of completeness we recall the Banach–Stone theorem and some other definitions from [9, Chapter I] which are needed for the vector-valued version.

THEOREM 2.1 ([2, Theorem 7.1]). *Let Ω be a locally compact Hausdorff space. If $T : C_0(\Omega) \to C_0(\Omega)$ is a surjective isometry, then there exist a homeomorphism $\phi : \Omega \to \Omega$ and a continuous map $u : \Omega \to \mathbb{T}$ such that*

$$Tf(\omega) = u(\omega)f(\phi(\omega)), \qquad \forall f \in C_0(\Omega),\, \omega \in \Omega.$$

Here, $\mathbb{T}$ denotes the unit circle in the complex plane.

DEFINITION 2.2. Let $T \in B(X)$.

(1) The operator T is called a *multiplier* of X if for every element $p \in \mathrm{ext}(B_{X^*})$, there exists $a_T(p) \in \mathbb{C}$ such that $T^*p = a_T(p)p$. The collection of all multipliers is denoted by $\mathrm{Mult}(X)$. Here, $\mathrm{ext}(B_{X^*})$ denotes the set of extreme points of B_{X^*}.

(2) The *centralizer* of X is defined as

$$Z(X) = \{T \in \mathrm{Mult}(X) : \exists \overline{T} \in \mathrm{Mult}(X)\ \forall p \in \mathrm{ext}(B_{X^*}),\ a_{\overline{T}}(p) = \overline{a_T(p)}\}.$$

DEFINITION 2.3. A Banach space X is said to have *trivial centralizer* if the dimension of $Z(X)$ is 1, that is, the only elements in the centralizer are scalar multiples of the identity operator I. This is obviously true if X is the scalar field.

THEOREM 2.4 ([2, Theorem 8.10]). *Let Ω be a locally compact Hausdorff space, and let X be a Banach space with trivial centralizer. If $T : C_0(\Omega, X) \to C_0(\Omega, X)$ is a surjective isometry, then there exist a homeomorphism $\phi : \Omega \to \Omega$ and a map $u : \Omega \to \mathcal{G}(X)$, continuous with respect to the strong operator topology of $B(X)$, such that*

$$Tf(\omega) = u_\omega(f(\phi(\omega))), \qquad \forall f \in C_0(\Omega, X),\ \omega \in \Omega.$$

For simplicity, we denote $u(\omega)$ by u_ω.

Definition 2.5 ([2, Definition 8.2]). A Banach space X is said to have the *strong Banach–Stone property* if it satisfies the conditions in Theorem 2.4.

It is known that strictly convex spaces have trivial centralizer. In particular, they have the strong Banach–Stone property.

Before we proceed let us see that ℓ_∞ is algebraically reflexive in $B(\ell_2)$.

Let $T \in \overline{\ell}^a_\infty$. Then, for each $f \in \ell_2$ we have $Tf(j) = \phi_f(j)f(j)$ for some $\phi_f \in \ell_\infty$. Hence, for the standard unit vectors e_j in ℓ_2,

$$Te_j(k) = \phi_{e_j}(k)e_j(k) = \begin{cases} \phi_{e_j}(j) & \text{for } j = k, \\ 0 & \text{for } j \neq k. \end{cases}$$

This implies that $Te_j = \phi_{e_j}(j)e_j$. Now, for $f \in \ell_2$ we have

$$Tf = T\Big(\sum_{j=1}^{\infty} f(j)e_j\Big) = \sum_{j=1}^{\infty} f(j)\phi_{e_j}(j)e_j = f\phi,$$

where $\phi = (\phi_{e_j}(j))$. As T is a bounded linear operator, $\phi \in \ell_\infty$. Therefore, $T \in \ell_\infty$.

We can actually show that ℓ_∞ is topologically reflexive.

The following lemma will be useful later.

Lemma 2.6. *Let $T \in \mathcal{G}(C_0(\Omega, X))$. Then T is an isometry of order n if and only if there exist a homeomorphism ϕ of Ω and a map $u : \Omega \to \mathcal{G}(X)$ satisfying*

$$u_\omega \circ u_{\phi(\omega)} \circ \cdots \circ u_{\phi^{n-1}(\omega)} = I, \qquad \phi^n(\omega) = \omega, \qquad \forall\, \omega \in \Omega,$$

where I denotes the identity map on X and T is given by

$$Tf(\omega) = u_\omega(f(\phi(\omega))), \qquad \forall\, f \in C_0(\Omega, X),\ \omega \in \Omega.$$

Proof. Since $T \in \mathcal{G}(C_0(\Omega, X))$, there exists a homeomorphism $\phi : \Omega \to \Omega$ and a map $u : \Omega \to \mathcal{G}(X)$ such that

$$Tf(\omega) = u_\omega(f(\phi(\omega))), \qquad \forall\, f \in C_0(\Omega, X),\ \omega \in \Omega.$$

As $T \in \mathcal{G}^n(C_0(\Omega, X))$ we have $T^n f(\omega) = f(\omega)$. This shows that

$$u_\omega \circ u_{\phi(\omega)} \circ \cdots \circ u_{\phi^{n-1}(\omega)}(f(\phi^n(\omega))) = f(\omega). \tag{2.1}$$

For a fixed $x \in X$ and $\omega \in \Omega$ consider the function $f = h \otimes x$, where $h \in C_0(\Omega)$ is such that $h(\omega) = h(\phi^n(\omega)) = 1$. Applying (2.1) to f we get

$$u_\omega \circ u_{\phi(\omega)} \circ \cdots \circ u_{\phi^{n-1}(\omega)}(x) = x.$$

Since this can be done for each $x \in X$ and each $\omega \in \Omega$, we conclude that

$$u_\omega \circ u_{\phi(\omega)} \circ \cdots \circ u_{\phi^{n-1}(\omega)} = I.$$

This also implies that $f(\phi^n(\omega)) = f(\omega)$ for all $f \in C_0(\Omega, X)$. Hence, $\phi^n(\omega) = \omega$. ∎

3. Algebraic reflexivity of $\mathcal{G}^n(C_0(\Omega,X))$. Our main result is the following.

THEOREM 3.1. *Let Ω be a first countable locally compact Hausdorff space, and let X be a Banach space which has the strong Banach–Stone property. If $\mathcal{G}(C_0(\Omega,X))$ is algebraically reflexive, then $\mathcal{G}^n(C_0(\Omega,X))$ is algebraically reflexive.*

Proof. Let $T \in \overline{\mathcal{G}^n(C_0(\Omega,X))}^a$. Then for each $f \in C_0(\Omega,X)$ we have $Tf(\omega) = u^f_\omega(f(\phi_f(\omega)))$ where $u^f : \Omega \to \mathcal{G}(X)$ is continuous in the strong operator topology and satisfies

$$u^f_\omega \circ u^f_{\phi_f(\omega)} \circ \cdots \circ u^f_{\phi_f^{n-1}(\omega)} = I,$$

and ϕ_f is a homeomorphism of Ω such that $\phi^n_f(\omega) = \omega$ for all $\omega \in \Omega$. In particular $T \in \overline{\mathcal{G}(C_0(\Omega,X))}^a$. Hence, there exist a homeomorphism $\phi : \Omega \to \Omega$ and a map $u : \Omega \to \mathcal{G}(X)$ such that

$$Tf(\omega) = u_\omega(f(\phi(\omega))), \quad \forall\, f \in C_0(\Omega,X),\, \omega \in \Omega.$$

To show that $T \in \mathcal{G}^n(C_0(\Omega,X))$, we need to prove that $T^n = I$, that is, by Lemma 2.6,

$$u_\omega \circ u_{\phi(\omega)} \circ \cdots \circ u_{\phi^{n-1}(\omega)} = I \quad \text{and} \quad \phi^n(\omega) = \omega, \quad \forall\, \omega \in \Omega.$$

Suppose $f = h \otimes x$, where h is a strictly positive function in $C_0(\Omega)$ and $0 \neq x \in X$. Then we have $Tf(\omega) = u^f_\omega(f(\phi_f(\omega))) = u_\omega(f(\phi(\omega)))$, that is, $u^f_\omega(h(\phi_f(\omega))x) = u_\omega(h(\phi(\omega))x)$. Taking norms on both sides and using the fact that u^f_ω, u_ω are isometries, and h is strictly positive, we get $u^f_\omega(x) = u_\omega(x)$. Therefore, $u^f_\omega = u_\omega$ for all $\omega \in \Omega$.

Let ω be any point in Ω. We consider the following cases.

CASE I: $\omega = \phi(\omega)$. Then

$$\phi^n(\omega) = \phi(\phi(\cdots(\phi(\omega))\cdots)) \ (n \text{ times}) = \omega.$$

We choose $h \in C_0(\Omega)$ such that $0 < h \leq 1$ and $h^{-1}\{1\} = \{\omega\}$. For $f = h \otimes x$, $0 \neq x \in X$, evaluating Tf at ω we get

$$\begin{aligned}
Tf(\omega) = u_\omega(f(\phi(\omega))) = u^f_\omega(f(\phi_f(\omega)))&\\
&\Longrightarrow u_\omega(h(\phi(\omega))x) = u^f_\omega(h(\phi_f(\omega))x)\\
&\Longrightarrow u_\omega(x) = u^f_\omega(h(\phi_f(\omega))x) \ \ (\text{as } h(\phi(\omega)) = h(\omega) = 1)\\
&\Longrightarrow \|u_\omega(x)\| = \|u^f_\omega(h(\phi_f(\omega))x)\|\\
&\Longrightarrow h(\phi_f(\omega)) = 1 \quad (u_\omega \text{ and } u^f_\omega \text{ are isometries})\\
&\Longrightarrow \phi_f(\omega) = \omega \quad (\text{by the choice of } h)\\
&\Longrightarrow \phi^2_f(\omega) = \cdots = \phi^{n-1}_f(\omega) = \omega.
\end{aligned}$$

So, we have

$$\begin{aligned} I &= u^f_\omega \circ u^f_{\phi_f(\omega)} \circ \cdots \circ u^f_{\phi_f^{n-1}(\omega)} \\ &= u^f_\omega \circ u^f_\omega \circ \cdots \circ u^f_\omega \\ &= u_\omega \circ u_\omega \circ \cdots \circ u_\omega \text{ (as } u^f_\omega = u_\omega). \end{aligned}$$

Case II: $\phi(\omega) \neq \omega$, $\phi^m(\omega) = \omega$ for some m that divides n and $\phi^s(\omega) \neq \omega$ for all $s < m$. As m divides n, there exists a positive integer q such that $n = mq$. Therefore,

$$\phi^n(\omega) = \phi^{mq}(\omega) = \phi^m(\phi^m(\cdots(\phi^m(\omega)))\cdots) \text{ (q times)} = \omega.$$

We now choose $h \in C_0(\Omega)$ such that $1 \leq h \leq m$ and

$$h^{-1}\{1\} = \{\omega\}, \quad h^{-1}\{2\} = \{\phi(\omega)\}, \ \ldots, \ h^{-1}\{m\} = \{\phi^{m-1}(\omega)\}.$$

Let $f = h \otimes x$ for $0 \neq x \in X$. Evaluating Tf at $\omega, \phi(\omega), \ldots, \phi^{m-1}(\omega)$ and considering our choice of the function h we get $\phi^p_f(\omega) = \phi^p(\omega)$ for $1 \leq p \leq m$.

This implies that

$$\phi_f^{m+1}(\omega) = \phi_f(\phi_f^m(\omega)) = \phi_f(\omega) = \phi(\omega) = \phi(\phi^m(\omega)) = \phi^{m+1}(\omega).$$

Thus, $\phi^p_f(\omega) = \phi^p(\omega)$ for $m+1 \leq p \leq n-1$. Since $u_\omega = u^f_\omega$ for all $\omega \in \Omega$, we have

$$u_\omega \circ u_{\phi(\omega)} \circ \cdots \circ u_{\phi^{n-1}(\omega)} = u^f_\omega \circ u^f_{\phi_f(\omega)} \circ \cdots \circ u^f_{\phi_f^{n-1}(\omega)} = I.$$

Case III: $\phi(\omega) \neq \omega$, $\phi^m(\omega) = \omega$ for some m that does not divide n and $\phi^s(\omega) \neq \omega$ for all $s < m$. Then there exist integers r and q such that $n = mq + r$, $0 < r < m$. We choose $h \in C_0(\Omega)$ such that $1 \leq h \leq m$ and

$$h^{-1}\{1\} = \{\omega\}, \quad h^{-1}\{2\} = \{\phi(\omega)\}, \ \ldots, \ h^{-1}\{m\} = \{\phi^{m-1}(\omega)\}.$$

By applying Tf at $\omega, \phi(\omega), \ldots, \phi^{m-1}(\omega)$ and proceeding as in Case II we get $\phi^p_f(\omega) = \phi^p(\omega)$ for $1 \leq p \leq n-1$. We now see that

$$\begin{aligned} Tf(\phi^{n-1}(\omega)) = u_{\phi^{n-1}(\omega)}(f(\phi^n(\omega))) &= u^f_{\phi^{n-1}(\omega)}(f(\phi_f(\phi^{n-1}(\omega)))) \\ &\implies u_{\phi^{n-1}(\omega)}(h(\phi^n(\omega))x) = u^f_{\phi^{n-1}(\omega)}(h(\phi_f(\phi_f^{n-1}(\omega)))x) \\ &\implies u_{\phi^{n-1}(\omega)}(h(\phi^n(\omega))x) = u^f_{\phi^{n-1}(\omega)}(h(\phi_f^n(\omega))x) \\ &\implies u_{\phi^{n-1}(\omega)}(h(\phi^n(\omega))x) = u^f_{\phi^{n-1}(\omega)}(h(\omega)x) \text{ (as } \phi^n_f(\omega) = \omega) \\ &\implies u_{\phi^{n-1}(\omega)}(h(\phi^n(\omega))x) = u^f_{\phi^{n-1}(\omega)}(x) \ \text{ (as } h(\omega) = 1) \\ &\implies \|u_{\phi^{n-1}(\omega)}(h(\phi^n(\omega))x)\| = \|u^f_{\phi^{n-1}(\omega)}(x)\| \\ &\implies h(\phi^n(\omega)) = 1 \quad (u_{\phi^{n-1}(\omega)} \text{ and } u^f_{\phi^{n-1}(\omega)} \text{ are isometries}) \\ &\implies \phi^n(\omega) = \omega \quad \text{(by the choice of } h). \end{aligned}$$

But our assumption that $\phi^m(\omega) = \omega$ implies that $\phi^{mq}(\omega) = \omega$. Hence,

$$\omega = \phi^n(\omega) = \phi^{r+mq}(\omega) = \phi^r(\phi^{mq}(\omega)) = \phi^r(\omega),$$

a contradiction because $r < m$.

CASE IV: $\omega, \phi(\omega), \ldots, \phi^{n-1}(\omega)$ are all distinct. Then choose $h \in C_0(\Omega)$ such that $1 \le h \le n$ and

$$h^{-1}\{1\} = \{\omega\}, \quad h^{-1}\{2\} = \{\phi(\omega)\}, \ \ldots, \ h^{-1}\{n\} = \{\phi^{n-1}(\omega)\}.$$

Proceeding as in Case III we get

$$\phi^n(\omega) = \omega \quad \text{and} \quad u_\omega \circ u_{\phi(\omega)} \circ \cdots \circ u_{\phi^{n-1}(\omega)} = I.$$

This completes the proof. ∎

COROLLARY 3.2. *Let Ω be a first countable locally compact Hausdorff space. Let X be a Banach space which has the strong Banach–Stone property, and does not have any generalized bi-circular projections. If $\mathcal{G}(C_0(\Omega, X))$ is algebraically reflexive, then the set $\mathcal{P}$ of generalized bi-circular projections on $C_0(\Omega, X)$ is also algebraically reflexive.*

Proof. Let $P \in \overline{\mathcal{P}}^a$. Then for each $f \in C_0(\Omega, X)$, there exists $P_f \in \mathcal{P}$ such that $Pf = P_f f$. Therefore, by [1, Theorem 4.2] and the assumption on X, for each f there exists a homeomorphism ϕ_f of Ω and $u^f : \Omega \to \mathcal{G}(X)$ satisfying

$$\phi_f^2(\omega) = \omega \quad \text{and} \quad u_\omega^f \circ u_{\phi_f(\omega)}^f = I, \qquad \forall\, \omega \in \Omega,$$

such that

$$Pf(\omega) = \tfrac{1}{2}[f(\omega) + u_\omega^f(f(\phi_f(\omega)))].$$

Therefore, for each $f \in C_0(\Omega, X)$, we get $(2P - I)f(\omega) = u_\omega^f(f(\phi_f(\omega)))$. This implies that $2P - I \in \overline{\mathcal{G}^2(C_0(\Omega, X))}^a$. The conclusion follows from Theorem 3.1. ∎

Combining Theorem 3.1 with [12, Theorem 7] we have the following corollary.

COROLLARY 3.3. *Let Ω be a first countable compact Hausdorff space, and let X be a uniformly convex Banach space such that $\mathcal{G}(X)$ is algebraically reflexive. Then $\mathcal{G}^n(C(\Omega, X))$ is algebraically reflexive.*

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