.

GENERALIZED 3-CIRCULAR PROJECTIONS FOR UNITARY CONGRUENCE INVARIANT NORMS

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Abstract. A projection P_0 on a complex Banach space is generalized 3-circular if its linear combination with two projections P_1 and P_2 having coefficients λ_1 and λ_2, respectively is a surjective isometry, where λ_1 and λ_2 are distinct unit modulus complex numbers different from 1 and $P_0 \oplus P_1 \oplus P_2 = I$. Such projections are always contractive. In this paper, we prove structure theorems for generalized 3-circular projections acting on the spaces of all $n \times n$ symmetric and skew-symmetric matrices over $\mathbb{C}$ when these spaces are equipped with unitary congruence invariant norms.

1. Introduction

The study of projections on Banach spaces is of great interest since they appear as building blocks of more complicated operators. This is clearly demonstrated by the powerful spectral Theory of Operators. Furthermore, spaces supporting a rich collection of projections present a very nice structure as, for example, the von Neumann algebras.

A class of projections, known as the *generalized bi-circular projections* (henceforth *GBP*), has recently attracted the attention of many mathematicians. This class was introduced by Fošner, Ilišević, and Li [9] in 2007. A projection P on a Banach space X is said to be a GBP if $P + \lambda(I - P)$ is a surjective isometry on X, where $\lambda \in \mathbb{T} \setminus \{1\}$. Here, $\mathbb{T}$ denotes the unit circle in the complex plane. In [9], the authors characterized $GBPs$ on finite-dimensional Banach spaces with respect to various G-invariant norms. Descriptions of $GBPs$ for different Banach spaces can be found in [1, 5, 11], and [12].

GBPs are one of the generalizations of the notion of orthogonal projections from Hilbert spaces to arbitrary Banach spaces. To be precise, if $\mathcal{H}$ is a Hilbert space, then P is a GBP on $\mathcal{H}$ if and only if P is an orthogonal projection, (see [7, Proposition 3.1]).

Moreover, it was shown in [17] that $GBPs$ are bicontractive. We say a projection P is *contractive* (resp., *bicontractive*) if $\|P\| = 1$ (resp., $\|P\| = \|I - P\| = 1$). Attempts to describe the structure of contractive or bicontractive projections on classical Banach spaces like $C_0(\Omega)$ or L_p and on spaces of operators, especially C^*- algebras, have received lots of attention in past as well as in recent time. The seminal work by Lindenstrauss [18] and the book [13] by Lacey are two classical references for the study of contractive projections.

2010 *Mathematics Subject Classification.* Primary 46B20; Secondary 47L05.

Key words and phrases. Spectral theorem, Isometry, Generalized 3-circular projection, Unitary congruence invariant norm.

Furthermore, it has been shown by Benau, Lacey [4], Dutta, Rao [8] and Lima [16] that on certain function spaces, for any bicontractive projection P, $\Phi = 2P - I$ is an isometry, which implies that P is a GBP. These include the spaces L_p $(1 \leq p < \infty)$, $C(\Omega)$, $C(\Omega, X)$ and the space of affine continuous functions on a Choquet simplex, $A(K)$.

The notion of a GBP was generalized in [2, 3] as follows.

Definition 1.1. Let X be a complex Banach space. A projection P_0 on X is said to be a *generalized n-circular projection*, (GnP, for short), $n \geq 2$, if there exist $\lambda_1, \lambda_2, \ldots, \lambda_{n-1} \in \mathbb{T} \setminus \{1\}$, λ_i, $i = 1, 2, \ldots, n-1$ of finite order and nontrivial projections $P_1, P_2, \ldots, P_{n-1}$ on X such that

(a) $\lambda_i \neq \lambda_j$ for $i \neq j$
(b) $P_0 \oplus P_1 \oplus \cdots \oplus P_{n-1} = I$
(c) $P_0 + \lambda_1 P_1 + \cdots + \lambda_{n-1} P_{n-1}$ is a surjective isometry.

Recently, in [2] the authors studied generalized 3-circular projections ($G3Ps$ for short) on $C(\Omega)$, where Ω is a compact connected Hausdorff space. Let P_0 be a $G3P$ on $C(\Omega)$; that is, $P_0 + \lambda_1 P_1 + \lambda_2 P_2 = T$ for some surjective isometry T, and λ_i and P_i are as in Definition 1.1, $i = 1,\ 2$. Then it was shown that λ_1 and λ_2 are cube roots of unity and $P_0 = \frac{I+T+T^2}{3}$ such that $T^3 = I$. The main reason for this characterization is the fact that $GBPs$ on $C(\Omega)$ are of the form $\frac{I+T}{2}$, where $T^2 = I$ (see [6]). This raises the question of whether $G3Ps$ on other Banach spaces are of the above form when $GBPs$ are of the form $\frac{I+L}{2}$, where L is a surjective isometry and $L^2 = I$. The obvious candidates for investigation are finite-dimensional Banach spaces like $\mathbb{C}^n$ or space of matrices. If X is an n-dimensional inner product space and $\|\cdot\|$ is a norm on X, which is a multiple of the norm induced by the inner product, then any GBP is an orthogonal projection (see [9, Proposition 2.1]); hence, we have to consider other norms like symmetric norms or unitary congruence invariant norms.

Now, different norms on finite- or infinite-dimensional Banach spaces are useful in many geometrical and analytical problems. The expository article by Chi-Kwong Li [15] is a pertinent reference for the importance of studying different kinds of norms.

The structures of $G3Ps$ on $\mathbb{C}^n$ and $\mathbb{M}_{m\times n}(\mathbb{C})$, where these spaces are equipped with a symmetric norm, are described in [3]. The purpose of this paper is to give complete descriptions of the structures of $G3Ps$ on the spaces of symmetric and skew-symmetric matrices when these spaces are equipped with a unitary congruence invariant norm.

2. Preliminaries and notation

Given two matrices $A,\ B \in \mathbb{M}_n(\mathbb{C})$, A is said to be *unitarily similar* to B if there exists a unitary $U \in \mathbb{M}_n(\mathbb{C})$ such that $A = U^*BU$. Similarly, A is said to be *unitarily congruent* to B if $A = U^tBU$ for some unitary $U \in \mathbb{M}_n(\mathbb{C})$. Unitary similarity is a natural equivalence relation in the study of normal or Hermitian matrices: U^*AU is normal (resp., Hermitian) if U is unitary and A is normal (resp., Hermitian). Unitary congruence is a natural equivalence relation in the study of complex symmetric or skew-symmetric matrices: U^tAU is symmetric (respectively, skew symmetric) if U is unitary and A is symmetric (respectively, skew symmetric). We refer the reader to [10] for more details on this subject.

Let us denote by

$S_n(\mathbb{C})$: the space of all $n \times n$ symmetric matrices over $\mathbb{C}$,

$K_n(\mathbb{C})$: the space of all $n \times n$ skew-symmetric matrices over $\mathbb{C}$, and

$U(\mathbb{C}^n)$: the group of all unitary operators on $\mathbb{C}^n$.

We recall the definition of a unitary congruence invariant norm.

Definition 2.1. A norm on $X = S_n(\mathbb{C})$ or $K_n(\mathbb{C})$ is called *unitary congruence invariant* if for every $A \in X$ we have $\|U^tAU\| = \|A\|$ for all $U \in U(\mathbb{C}^n)$.

To characterize $G3Ps$, we first need to identify the surjective linear isometries on $S_n(\mathbb{C})$ and $K_n(\mathbb{C})$ for unitary congruence invariant norms. The descriptions of the isometry group of these spaces are given in the following theorems.

Theorem 2.2. [14, Theorem 2.8] *For a unitary congruence invariant norm on $S_n(\mathbb{C})$, which is not a multiple of the Frobenius norm, any isometry T is given by $T(A) = U^tAU$, where $U \in U(\mathbb{C}^n)$.*

Theorem 2.3. [14, Theorem 2.9] *For a unitary congruence invariant norm on $K_n(\mathbb{C})$, $n \neq 4$, which is not a multiple of the Frobenius norm, any isometry T is given by $T(A) = U^tAU$, where $U \in U(\mathbb{C}^n)$.*

If $n = 4$, then any isometry T is given by either $T(A) = U^tAU$ or $T(A) = \psi(U^tAU)$, where $U \in U(\mathbb{C}^n)$ and $\psi(A)$ is obtained from A by interchanging its $(1,4)$ and $(2,3)$ entries, and interchanging its $(4,1)$ and $(3,2)$ entries.

Remark 2.4. In the sequel, whenever we mention that P_0 is a $G3P$ and write $P_0 + \lambda_1 P_1 + \lambda_2 P_2 = T$, we will always mean that T, λ_i, and P_i, $i = 1,2$ are as in Definition 1.1. The scalars λ_1 and λ_2 will be sometimes referred to as the scalars associated with P_0.

Remark 2.5. *Let P_0 be a $G3P$ on a Banach space X such that $P_0 + \lambda_1 P_1 + \lambda_2 P_2 = T$. Then*

$$P_0 = \frac{(T - \lambda_1 I)(T - \lambda_2 I)}{(1 - \lambda_1)(1 - \lambda_2)}, \quad P_1 = \frac{(T - I)(T - \lambda_2 I)}{(\lambda_1 - 1)(\lambda_1 - \lambda_2)}$$

and

$$P_2 = \frac{(T - I)(T - \lambda_1 I)}{(\lambda_2 - 1)(\lambda_2 - \lambda_1)}.$$

The following Lemma will be useful later. Its proof is similar to the proof of Lemma 2.1 in [2].

Lemma 2.6. *Let X be a Banach space satisfying the following property:*

whenever P is a projection on X such that $P + \lambda(I - P)$ is a surjective isometry, we have $\lambda = -1$.

Let P_0 be a $G3P$ on X such that $P_0 + \lambda_1 P_1 + \lambda_2 P_2 = T$. Then λ_1 and λ_2 are of the same order.

3. Structure of $G3P$ for symmetric matrices

In this section we characterize $G3Ps$ on $S_n(\mathbb{C})$ with a unitary congruence invariant norm.

Remark 3.1. *Suppose that $T : S_n(\mathbb{C}) \longrightarrow S_n(\mathbb{C})$ is defined by $T(A) = U^tAU$, where $U \in U(\mathbb{C}^n)$. Assume that U^t has eigenvalues $\mu_1, \mu_2, \dots, \mu_n$ with eigenvectors $x_1, x_2, \dots, x_n$. Then T has eigenvalues $\mu_i\mu_j$ with eigenvectors $x_ix_j^t + x_jx_i^t$ for $1 \leq i, j \leq n$. To see this, observe that, for any two eigenvalues μ_i and μ_j of U^t with corresponding eigenvectors x_i and x_j, we have*

$$\begin{aligned} T(x_ix_j^t + x_jx_i^t) &= U^t(x_ix_j^t + x_jx_i^t)U \\ &= U^tx_ix_j^tU + U^tx_jx_i^tU \\ &= \mu_ix_i\mu_jx_j^t + \mu_jx_j\mu_ix_i^t \\ &= \mu_i\mu_j(x_ix_j^t + x_jx_i^t). \end{aligned}$$

Now, if λ is an eigenvalue of T with eigenvector A, then $U^tAU = \lambda A$ or $U^tA = \lambda AU^*$. For an eigenvalue μ_i of U^t with eigenvector x_i, we have $U^tA\overline{x_i} = \lambda AU^*\overline{x_i} = \lambda A\overline{\mu_ix_i} = \lambda\overline{\mu_i}A\overline{x_i}$. This implies that $\lambda\overline{\mu_i}$ is an eigenvalue of U^t, and hence $\lambda\overline{\mu_i} = \mu_j$ for some j. As eigenvalues of a unitary matrix are of a unit modulus, we have $\lambda = \mu_i\mu_j$ or $\lambda = \mu_i^2$ if $i = j$.

Theorem 3.2. *Let $\|\cdot\|$ be a unitary congruence invariant norm on $S_n(\mathbb{C})$, which is not a multiple of the Frobenius norm, and let P_0 a $G3P$. Then there exist an integer p and $R_i = R_i^* = R_i^2$ in $\mathbb{M}_n(\mathbb{C})$ such that*

$$P_0(A) = \sum_{i=0}^{p-1} R_i^tAR_{(p-i)(mod\ p)},$$

where

(i) $i = 0, 1, \dots, p-1$ and p is an odd integer ≥ 3,
(ii) $R_iR_j = 0$ for $i \neq j$,
(iii) $\sum_{i=0}^{p-1} R_i = I$.

Proof. Let $P_0 + \lambda_1P_1 + \lambda_2P_2 = T$ such that T is of the form $A \longmapsto U^tAU$ for some $U \in U(\mathbb{C}^n)$. The spectrum of T is $\{1, \lambda_1, \lambda_2\}$. Suppose that U has eigenvalues $\mu_1, \mu_2, \dots, \mu_n$. Then T has eigenvalues $\mu_i\mu_j$, $1 \leq i, j \leq n$.

We claim that U can have two or three distinct eigenvalues.

To see the claim, suppose that U has one eigenvalue, say, μ. Then T will have eigenvalue μ^2, which is a contradiction.

If U has four distinct eigenvalues, say, μ_1, μ_2, μ_3, and μ_4, then $\mu_1\mu_2$, $\mu_1\mu_3$, $\mu_1\mu_4$ and μ_1^2 are distinct eigenvalues of T, which is impossible. Similarly, U cannot have more than four distinct eigenvalues.

So, we consider the following two steps.

Step I

Assume that μ_1, μ_2, and μ_3 are distinct eigenvalues of U. Then the set $A = \{\mu_1^2, \mu_1\mu_2, \mu_1\mu_3, \mu_2^2, \mu_2\mu_3, \mu_3^2\}$ consists of eigenvalues of T. The elements μ_1^2, $\mu_1\mu_2$,

$\mu_1\mu_3$ are all distinct. Therefore, $\mu_2\mu_3 = \mu_1^2$, which implies that $\mu_2^2 = \mu_1\mu_3$ and $\mu_3^2 = \mu_1\mu_2$. Then $A = \{\mu_1^2, \mu_2^2, \mu_3^2\}$. Due to the symmetry of these elements it is sufficient to consider $\mu_1^2 = 1$, $\mu_2^2 = \lambda_1$, and $\mu_3^2 = \lambda_2$. Thus, $\mu_1\mu_3 = \lambda_1$, $\mu_3^2 = \lambda_2 = \lambda_1^2$, and $\mu_2^2\mu_3^2 = 1 = \lambda_1\lambda_2$. Therefore, λ_1 and λ_2 are cube roots of unity, and hence $T^3(A) = A = X^tAX$ for all $A \in S_n(\mathbb{C})$, where $X = U^3$. Putting $A = I$, we have $X^tX = I$ or $X^t = X^{-1}$. This implies that $A = X^tAX = X^{-1}AX$ or $XA = AX$. But the centralizer of the space of symmetric matrices is $\pm I$, and so $X = I$ or $-I$.

Let $U^3 = I$. We put

$$R_i = \frac{I + \alpha_i U + \alpha_i^2 U^2}{3},$$

where $i = 0, 1, 2$; $\alpha_0 = 1, \alpha_1 = \omega$; and $\alpha_2 = \omega^2$. Then we have

$$P_0A = R_0^tAR_0 + R_1^tAR_2 + R_2^tAR_1.$$

Let $U^3 = -I$. We put

$$R_i = \frac{I - \alpha_i U + \alpha_i^2 U^2}{3},$$

where $i = 0, 1, 2$, $\alpha_0 = 1, \alpha_1 = \omega$; and $\alpha_2 = \omega^2$. Then we obtain

$$P_0A = R_0^tAR_0 + R_1^tAR_2 + R_2^tAR_1.$$

In both cases, it is straightforward to verify that $R_i = R_i^* = R_i^2$ for $i \neq j$, $R_iR_j = 0$, and $R_0 + R_1 + R_2 = I$; and hence, the theorem is proved for $p = 3$.

Step II

Suppose that U has two distinct eigenvalues, say, μ_1 and μ_2. Then the spectrum of T will be $\{\mu_1^2, \mu_2^2, \mu_1\mu_2\} = \{1, \lambda_1, \lambda_2\}$.

Lemma 2.6 and Proposition 5.1 in [9] imply that λ_1 and λ_2 have the same order. Let p be the order of λ_1.

Consider the following two cases:

(a) If $\mu_1^2 = 1$, $\mu_2^2 = \lambda_2$, and $\mu_1\mu_2 = \lambda_1$, then we get $\lambda_1^2 = \lambda_2$ or $\lambda_1 = \pm\sqrt{\lambda_2}$.

We first claim that $\lambda_1 \neq -\sqrt{\lambda_2}$. To see this, if $\lambda_1 = -\sqrt{\lambda_2}$, then we have $\lambda_1^p = (-\sqrt{\lambda_2})^p = 1$ or $(-1)^p(\lambda_2)^{p/2} = 1$. This shows that p is odd; otherwise, $(\lambda_2)^{p/2} = 1$, which is a contradiction because the order of λ_2 is p. Hence, we get $(\lambda_2)^{p/2} = -1$. It follows that $\lambda_1^p = -1$, a contradiction since the order of λ_1 is p.

Thus, we must have $\lambda_1 = \sqrt{\lambda_2}$ and $\lambda_1^p = (\sqrt{\lambda_2})^p = (\lambda_2)^{p/2} = 1$. This implies that p is odd. As the order of λ_1 is p, we have $U^p = I$. Further, for $i = 0, 1, \ldots, p-1$, we have

$$P_0 + \lambda_1^i P_1 + \lambda_2^i P_2 = T^i.$$

Adding these equations, we get

$$pP_0 + (\sum_{i=0}^{p-1} \lambda_1^i)P_1 + (\sum_{i=0}^{p-1} \lambda_2^i)P_2 = I + T + T^2 + \cdots + T^{p-1}.$$

Since $\sum_{i=0}^{p-1} \lambda_1^i = \sum_{i=0}^{p-1} \lambda_2^i = 0$, we obtain

$$P_0 = \frac{I + T + T^2 + \cdots + T^{p-1}}{p}.$$

We now define

$$R_i = \frac{1}{p} \sum_{j=0}^{p-1} \lambda_1^{ij} U^j,$$

where $i = 0, 1, \ldots, p-1$. It can be easily verified that $R_i = R_i^* = R_i^2$ for $i \neq j$, $R_i R_j = 0$, and $\sum_{i=0}^{p-1} R_i = I$.

Therefore, P_0 will be of the form

$$P_0(A) = \sum_{i=0}^{p-1} R_i^t A R_{(p-i)(mod\ p)}.$$

We can also get the form of P_1 and P_2. We first observe that P_j, $j = 1, 2$, will have the form

$$P_j = \frac{I + \overline{\lambda_j} T + \overline{\lambda_j}^2 T^2 + \cdots + \overline{\lambda_j}^{p-1} T^{p-1}}{p}.$$

But $\overline{\lambda_j} = \lambda_j^{p-1}$ and $\lambda_1^2 = \lambda_2$, so we get

$$P_1(A) = \sum_{i=0}^{p-1} R_i^t A R_{(p-1-i)(mod\ p)}.$$

Similarly,

$$P_2(A) = \sum_{i=0}^{p-1} R_i^t A R_{(p-2-i)(mod\ p)}.$$

Here, we note that the order of λ_1 and λ_2 can be 3.

(b) If $\mu_1^2 = \lambda_1$, $\mu_2^2 = \lambda_2$, and $\mu_1 \mu_2 = 1$, then we get $\lambda_1 \lambda_2 = 1$. Now,

$$\begin{aligned} T &= P_0 + \lambda_1 P_1 + \overline{\lambda_1} P_2 \\ \Longrightarrow \quad \lambda_1 T &= P_2 + \lambda_1 P_0 + \lambda_1^2 P_1. \end{aligned}$$

Because $\lambda_1 T$ is again an isometry, we are reduced to the previous case, and so P_2 will be of the form $P_2(A) = \sum_{i=0}^{p-1} R_i^t A R_{(p-i)(mod\ p)}$, where the R_i's satisfy conditions $(i) - (iii)$ of Theorem 3.2.

Proceeding in the same way as above, we can easily obtain the form of P_0.

This completes the proof. $\square$

4. Structure of $G3Ps$ for skew-symmetric matrices

In this section, we identify the structure of $G3Ps$ on $K_n(\mathbb{C})$ with a unitary congruence invariant norm.

Remark 4.1. *Suppose that $T : K_n(\mathbb{C}) \longrightarrow K_n(\mathbb{C})$ is defined by $T(A) = U^tAU$, where $U \in U(\mathbb{C}^n)$. Assume that U^t has eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$ with eigenvectors $x_1, x_2, \ldots, x_n$. Then, arguing in a similar fashion as we did in Remark 3.1, we can show that T has eigenvalues $\mu_i\mu_j$ with eigenvectors $x_ix_j^t - x_jx_i^t$ for $1 \leq i < j \leq n$. Now, suppose that μ_i is an eigenvalue of multiplicity at least 2 and x_i, y_i are the corresponding eigenvectors. In this case,*

$$\begin{aligned} T(x_iy_i^t - y_ix_i^t) &= U^t(x_iy_i^t - y_ix_i^t)U \\ &= U^tx_iy_i^tU - U^ty_ix_i^tU \\ &= \mu_ix_i\mu_iy_i^t - \mu_iy_i\mu_ix_i^t \\ &= \mu_i^2(x_iy_i^t - y_ix_i^t). \end{aligned}$$

Therefore, we conclude that μ_i^2 is an eigenvalue of T if the multiplicity of the eigenvalue μ_i is at least 2.

The following remark will be used in the proof of Theorem 4.3 (see the remark before Proposition 5.1 in [9]).

Remark 4.2. We note that the mapping on $K_4(\mathbb{C})$ defined by $A \longmapsto \psi(UAU^t)$ can be written as $A \longmapsto det(U)W\psi(A)W^t$ with $W = R\overline{U}R$, where $R = E_{14} - E_{23} + E_{32} - E_{41}$.

Since $K_2(\mathbb{C})$ is one-dimensional, we assume that $n \geq 3$.

Theorem 4.3. *Let $\|\cdot\|$ be a unitary congruence invariant norm on $K_n(\mathbb{C})$ not equal to a multiple of the Frobenius norm, $n \geq 3$ and P_0 be a $G3P$. Suppose the scalars λ_1 and λ_2 associated with P_0 are cube roots of unity. Then one and only one of the following assertions holds:*

(*a*) *There exist $R_i = R_i^* = R_i^2$ in $\mathbb{M}_n(\mathbb{C})$ such that $R_iR_j = 0$ for $i \neq j$, $R_0 + R_1 + R_2 = I$, and $P_0(A) = R_0^tAR_0 + R_1^tAR_2 + R_2^tAR_1$.*

(*b*) *$n = 4$ and the isometry associated with P_0 is of the form $A \longmapsto \psi(UAU^t)$. Then there exist $U \in U(\mathbb{C}^4)$, $\alpha, \beta \in \mathbb{C}$ with $\alpha^3 = \beta^2$, $\alpha = \frac{1}{det(U)}$, and $V \in U(\mathbb{C}^4)$ such that $\psi(U^tAU) = \alpha V^tAV$, $V^3 = \frac{1}{\beta}I$, and*

$$P_0(A) = \frac{A + \alpha V^tAV + \alpha^2(V^t)^2AV^2}{3}.$$

Proof. Let $P_0 + \lambda_1P_1 + \lambda_2P_2 = T$, where $T(A) = U^tAU$ for some $U \in U(\mathbb{C}^n)$. As λ_1 and λ_2 are cube roots of unity, we have $T^3 = I$. Thus, for all $A \in K_n(\mathbb{C})$, $A = T^3(A) = X^tAX$, where $X = U^3$. This is possible if and only if $X = I$ or $-I$. If $U^3 = I$, then we define

$$R_i = \frac{I + \alpha_iU + \alpha_i^2U^2}{3},\ i = 0, 1, 2,\ \alpha_0 = 1,\ \alpha_1 = \omega,\ \text{and } \alpha_2 = \omega^2.$$

It follows that

$$P_0A = R_0^tAR_0 + R_1^tAR_2 + R_2^tAR_1.$$

If $U^3 = -I$, we define

$$R_i = \frac{I - \alpha_i U + \alpha_i^2 U^2}{3}, \ i = 0,1,2, \ \alpha_0 = 1, \ \alpha_1 = \omega, \ \text{and } \alpha_2 = \omega^2.$$

We conclude that P_0 has the form

$$A \longmapsto R_0^t A R_0 + R_1^t A R_2 + R_2^t A R_1.$$

In both cases, it can be easily verified that $R_i = R_i^* = R_i^2$, $R_i R_j = 0$ for $i \neq j$ and $R_0 + R_1 + R_2 = I$.

Thus, we get assertion (a).

Suppose that $n = 4$ and that there is a $U \in U(\mathbb{C}^4)$ such that

$$T(A) = \psi(U^t A U) = det(U) W \psi(A) W^t,$$

with $W = R\overline{U}R$ and $R = E_{14} - E_{23} + E_{32} - E_{41}$. This implies that $\psi(T(A)) = \psi^2(U^t A U) = U^t A U$. Therefore,

$$\begin{aligned} T^2(A) &= T(T(A)) = \psi(U^t T(A) U) \\ &= det(U) W \psi(T(A)) W^t \\ &= det(U) W U^t A U W^t \\ &= det(U) X^t A X \text{ with } X = U W^t. \end{aligned}$$

It follows that

$$T^3(A) = T^2(T(A)) = det(U) X^t T(A) X.$$

Since $T^3 = I$ and $XX^* = X^*X = I$, we get $T(A) = \alpha \overline{X} A X^*$, where $\alpha = \frac{1}{det(U)}$. This implies that

$$T^2(A) = \alpha^2 \overline{X}^2 A (X^*)^2 \text{ and } T^3(A) = \alpha^3 \overline{X}^3 A (X^*)^3.$$

Since T^3 is the identity operator, there exists $\beta \in \mathbb{C}$ with $\beta^2 = \alpha^3$ such that $I = \beta (X^*)^3$.

Hence, assertion (b) is proved. □

Theorem 4.4. *Let $\|\cdot\|$ be a unitary congruence invariant norm on $K_n(\mathbb{C})$ not equal to a multiple of the Frobenius norm, $n \geq 3$ and P_0 be a G3P. Suppose the scalars λ_1 and λ_2 associated with P_0 are not cube roots of unity and $n \neq 4$. Then there exist $R_i = R_i^* = R_i^2$ in $\mathbb{M}_n(\mathbb{C})$, $i = 1, \ldots, p$ with $R_i R_j = 0$ for $i \neq j$, $R_i^t A R_i = 0$ for all $A \in K_n(\mathbb{C})$, and $U \in U(\mathbb{C}^n)$ such that one and only one of the following assertions holds:*

(a) *U has three distinct eigenvalues and each has multiplicity one and* $P_0(A) = A - (AR_1 + AR_2) + (AR_1 + AR_2)^t + 2(R_1^t A R_2 + R_2^t A R_1)$.

(b) *U has two distinct eigenvalues and $P_0(A)$ is equal to one of the following:*

(i) $\sum_{i=1}^{p} (AR_i + R_i^t A) - 2 \sum_{\substack{i,j=1 \\ i \neq j}}^{p} R_i^t A R_j$;

(ii) $\sum_{\substack{i,j=1 \\ i \neq j}}^{p} R_i^t A R_j$.

(c) *U has three distinct eigenvalues and only one has multiplicity greater than* 1*. Then $P_0(A)$ is equal to one of the following:*
(i) $AR_1 + R_1^t A - R_1^t AR_2 - R_2^t AR_1$*;*
(ii) $A - (AR_1 + AR_2) + (AR_1 + AR_2)^t + 2(R_1^t AR_2 + R_2^t AR_1)$*.*

Remark 4.5. In the case when $n = 4$, we were not able to find the structure of the $G3P$ P_0. Note that if P is GBP on $\mathbb{K}_4(\mathbb{F})$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, then the scalar associated with P is -1 (see Proposition 5.2 in [9]).

Since the proof of the above theorem is long, we divide it into lemmas and propositions.

Let $P_0 + \lambda_1 P_1 + \lambda_2 P_2 = T$, where $T(A) = U^t AU$ for some $U \in U(\mathbb{C}^n)$. Suppose that U has m distinct eigenvalues, say, $\mu_1, \mu_2, \ldots, \mu_m$.

We will first prove that the unitary matrix U has two or three distinct eigenvalues. If U has three distinct eigenvalues, then only one can have multiplicity greater than 1. In all the possible cases we will identify the structure of the $G3Ps$ P_0. As we will see later, we use the spectral theorem for normal matrices, which states that any normal matrix A is unitary diagonalizable; that is, there exist a $W \in U(\mathbb{C}^n)$ such that $A = W^* DW$, where D is a diagonal matrix.

Let us set some notation. Let $\{\mu_1, \mu_2, \ldots, \mu_k\}$ ($k \leq n$ and $\mu_i \neq \mu_j$ with $i \neq j$) be the eigenvalues of U with multiplicities $n_1, \ldots, n_k$ ($n_i \geq 1$) respectively. Remark 4.1 states that $\mu_i\mu_j$ $(i \neq j)$ is an eigenvalue of T. We observe that $k > 1$ since otherwise $U = \mu I$ and $T = \mu^2 I$. We also observe that if $k = 2$, then $n_i \geq 2$ for $i = 1, 2$.

Lemma 4.6. *If $\mu_1, \mu_2, \ldots, \mu_k$ are k distinct eigenvalues of U, then $k = 2$ or $k = 3$.*

Proof. Suppose $k \geq 4$. Then $\mu_1, \mu_2, \mu_3, \mu_4$ are all distinct. We have that $\mu_1\mu_2$, $\mu_1\mu_3$, $\mu_1\mu_4$ are also distinct and eigenvalues of T. This implies that

$$\mu_2\mu_3 = \mu_1\mu_4,\ \ \mu_2\mu_4 = \mu_1\mu_3 \text{ and } \mu_3\mu_4 = \mu_1\mu_2.$$

Therefore,

$$\mu_2\mu_3^2 = \mu_1\mu_3\mu_4 = \mu_2\mu_4^2 \text{ and } \mu_3 = -\mu_4.$$

Further, $\mu_3^2\mu_4 = \mu_2^2\mu_4$ implying that $\mu_3 = -\mu_2$. This leads to an absurdity since $\mu_2 \neq \mu_4$. This shows that $k \leq 3$ and completes the proof. □

Lemma 4.7. *If $k = 3$, then the unitary matrix U can have only one eigenvalue with multiplicity greater than* 1*.*

Proof. Suppose otherwise that μ_1, μ_2 and μ_3 are eigenvalues of U such that $n_i > 1$ $\forall\ i = 1, 2, 3$. Then the set $A = \{\mu_1^2, \mu_1\mu_2, \mu_1\mu_3, \mu_2^2, \mu_2\mu_3, \mu_3^2\}$ consists of eigenvalues of T. Proceeding exactly as in Step I of Theorem 3.2 we have that λ_1 and λ_2 are cube roots of unity. This is impossible since $\lambda_1 + \lambda_2 \neq -1$.

Now, suppose that $n_i > 1$ for $i = 1, 2$, then $A = \{\mu_1^2, \mu_1\mu_2, \mu_1\mu_3, \mu_2^2, \mu_2\mu_3\}$ and $\mu_1^2 = \mu_2\mu_3$, $\mu_2^2 = \mu_1\mu_3$. This implies that $\mu_1^2\mu_2^2 = \mu_1\mu_2\mu_3^2$ or $\mu_3^2 = \mu_1\mu_2$, and we are back to the previous case. This completes the proof. □

Now, we find the structure of P_0 in all the possible cases.

Proposition 4.8. *With the assumptions of Theorem 4.4, suppose that the unitary matrix U has three distinct eigenvalues each with multiplicity 1. Then there exist $R_i = R_i^* = R_i^2$ in $\mathbb{M}_n(\mathbb{C})$, $i = 1, 2$ with $R_iR_j = 0$ for $i \neq j$ and $R_i^tAR_i = 0$ for all $A \in K_n(\mathbb{C})$ such that*
$$P_0(A) = A - (AR_1 + AR_2) + (AR_1 + AR_2)^t + 2(R_1^tAR_2 + R_2^tAR_1).$$

Proof. Suppose that μ_1, μ_2 and μ_3 are the eigenvalues of U. Since $n_i = 1$ $\forall$ $i = 1, 2, 3$ we have $n = 3$. Moreover, the spectrum of T will be $\{\mu_1\mu_2, \mu_1\mu_3, \mu_2\mu_3\}$ which is equal to $\{1, \lambda_1, \lambda_2\}$.

Without loss of generality, we can assume that $\mu_1\mu_2 = 1, \mu_1\mu_3 = \lambda_1$ and $\mu_2\mu_3 = \lambda_2$. By the spectral theorem for normal matrices there exists a unitary matrix W such that
$$\begin{aligned} U &= W^* \operatorname{diag}(\mu_1, \mu_2, \mu_3)W = W^*(\mu_1E_{11} + \mu_2E_{22} + \mu_3E_{33})W \\ &= W^*(\mu_1E_{11} + \mu_2E_{22} + \mu_3(I - E_{11} - E_{22}))W. \end{aligned}$$
Let $R_i = W^*E_{ii}W$, $i = 1, 2$. Then we have $R_i^t = W^tE_{ii}\overline{W}$. This implies that
$$U = \mu_3I + (\mu_1 - \mu_3)R_1 + (\mu_2 - \mu_3)R_2.$$
We observe that $E_{ii}AE_{ii} = 0$ for all $A \in K_n(\mathbb{C})$ and hence we get $R_1^tAR_1 = R_2^tAR_2 = 0$. Now, we have
$$T(A) = U^tAU =$$
$$\begin{aligned} &[\mu_3A + (\mu_1 - \mu_3)R_1^tA + (\mu_2 - \mu_3)R_2^tA][\mu_3I + (\mu_1 - \mu_3)R_1 + (\mu_2 - \mu_3)R_2] \\ &= \mu_3^2A + \mu_3(\mu_1 - \mu_3)(AR_1 + R_1^tA) + \mu_3(\mu_2 - \mu_3)(AR_2 + R_2^tA) \\ &\quad + (\mu_1 - \mu_3)(\mu_2 - \mu_3)(R_1^tAR_2 + R_2^tAR_1). \\ &= \lambda_1\lambda_2A + \lambda_1(1 - \lambda_2)(AR_1 + R_1^tA) + \lambda_2(1 - \lambda_1)(AR_2 + R_2^tA) \\ &\quad + (1 - \lambda_1)(1 - \lambda_2)(R_1^tAR_2 + R_2^tAR_1). \end{aligned}$$
Similarly, we can show that
$$\begin{aligned} T^2(A) &= \mu_3^4A + \mu_3^2(\mu_1^2 - \mu_3^2)(AR_1 + R_1^tA) + \mu_3^2(\mu_2^2 - \mu_3^2)(AR_2 + R_2^tA) + \\ &\quad (\mu_1^2 - \mu_3^2)(\mu_2^2 - \mu_3^2)(R_1^tAR_2 + R_2^tAR_1) \\ &= \lambda_1^2\lambda_2^2A + \lambda_1^2(1 - \lambda_2^2)(AR_1 + R_1^tA) + \lambda_2^2(1 - \lambda_1^2)(AR_2 + R_2^tA) \\ &\quad + (1 - \lambda_1^2)(1 - \lambda_2^2)(R_1^tAR_2 + R_2^tAR_1). \end{aligned}$$
Therefore, we have
$$P_0(A) = \frac{T^2(A) - (\lambda_1 + \lambda_2)T(A) + \lambda_1\lambda_2A}{(1 - \lambda_1)(1 - \lambda_2)}$$
$$= \lambda_1\lambda_2[A - AR_1 - R_1^tA - AR_2 - R_2^tA] + (1 + \lambda_1\lambda_2)(R_1^tAR_2 + R_2^tAR_1).$$
Computing $P_0^2(A)$ and using the fact that P_0 is a projection, we get $\lambda_1\lambda_2 = 1$. Therefore,
$$P_0(A) = A - (AR_1 + AR_2) + (AR_1 + AR_2)^t + 2(R_1^tAR_2 + R_2^tAR_1).$$
This completes the proof of assertion (a) of Theorem 4.4 $\square$

Proposition 4.9. *With the assumptions of Theorem 4.4, suppose that U has two distinct eigenvalues. Then $P_0(A)$ is equal to one of the following:*

(a) $\sum_{i=1}^{p}(AR_i + R_i^t A) - 2\sum_{\substack{i,j=1\\ i\neq j}}^{p} R_i^t A R_j$;

(b) $\sum_{\substack{i,j=1\\ i\neq j}}^{p} R_i^t A R_j$.

Proof. Suppose that μ_1 and μ_2 are the two distinct eigenvalues of U with $n_i \geq 2$. Thus, the spectrum of T is $\{\mu_1^2, \mu_2^2, \mu_1\mu_2\} = \{1, \lambda_1, \lambda_2\}$.

If $\mu_1^2 = \lambda_1$, $\mu_2^2 = \lambda_2$ and $\mu_1\mu_2 = 1$, then we get $\lambda_1\lambda_2 = 1$.

If $\mu_1^2 = 1$, $\mu_2^2 = \lambda_2$ and $\mu_1\mu_2 = \lambda_1$, then we get $\lambda_2 = \lambda_1^2$.

Consequently the eigenvalues of U will have one of the following patterns

(a) $\sqrt{\lambda_1}, \sqrt{\lambda_1}, \ldots, \sqrt{\lambda_1}, \sqrt{\lambda_2}, \sqrt{\lambda_2}, \ldots, \sqrt{\lambda_2}$,

(b) $-\sqrt{\lambda_1}, -\sqrt{\lambda_1}, \ldots, -\sqrt{\lambda_1}, -\sqrt{\lambda_2}, -\sqrt{\lambda_2}, \ldots, -\sqrt{\lambda_2}$,

(c) $1, 1, \ldots, 1, \lambda_1, \lambda_1, \ldots, \lambda_1$ or

(d) $-1, -1, \ldots, -1, -\lambda_1, -\lambda_1, \ldots, -\lambda_1$.

Now, there exists a unitary matrix W such that

$$U = W^* \operatorname{diag}(\mu_1, \ldots, \mu_1, \mu_2, \ldots, \mu_2)W.$$

Suppose the multiplicities of μ_1 and μ_2 are are p and q respectively. Then we have

$$\begin{aligned} U &= W^*(\mu_1 E_{11} + \cdots + \mu_1 E_{pp} + \mu_2 E_{p+1p+1} + \cdots + \mu_2 E_{nn})W \\ &= W^*(\mu_1 E_{11} + \cdots + \mu_1 E_{pp} + \mu_2(I - E_{11} - \cdots - E_{pp}))W \\ &= \mu_2 I + (\mu_1 - \mu_2)W^*(E_{11} + \cdots + E_{pp})W \end{aligned}$$

Let $R_i = W^* E_{ii} W$, $i = 1, \ldots, p$ so that we get

$$U = \mu_2 I + (\mu_1 - \mu_2)(R_1 + \cdots + R_p).$$

As we observed earlier, $R_i^t A R_i = 0$ for all $A \in K_n(\mathbb{C})$. Consequently, we have

$$\begin{aligned} &T(A) = U^t A U \\ &= [\mu_2 A + (\mu_1 - \mu_2)(R_1^t A + \cdots + R_p^t A)][\mu_2 I + (\mu_1 - \mu_2)(R_1 + \cdots + R_p)] \\ &= \mu_2^2 A + \mu_2(\mu_1 - \mu_2)\sum_{i=1}^{p}(AR_i + R_i^t A) + (\mu_1 - \mu_2)^2 \sum_{\substack{i,j=1\\ i\neq j}}^{p} R_i^t A R_j. \end{aligned}$$

Similarly, we have

$$\begin{aligned} &T^2(A) \\ &= [\mu_2^2 A + (\mu_1^2 - \mu_2^2)(R_1^t A + \cdots + R_p^t A)][\mu_2^2 I + (\mu_1^2 - \mu_2^2)(R_1 + \cdots + R_p)] \\ &= \mu_2^4 A + \mu_2^2(\mu_1^2 - \mu_2^2)\sum_{i=1}^{p}(AR_i + R_i^t A) + (\mu_1^2 - \mu_2^2)^2 \sum_{\substack{i,j=1\\ i\neq j}}^{p} R_i^t A R_j. \end{aligned}$$

If (a) or (b) holds, then the expressions of $T(A)$ and $T^2(A)$ become

$$T(A) = \lambda_2 A + (1 - \lambda_2)\sum_{i=1}^{p}(AR_i + R_i^t A) + (\lambda_1 + \lambda_2 - 2)\sum_{\substack{i,j=1\\ i\neq j}}^{p} R_i^t A R_j,$$

$$T^2(A) = \lambda_2^2 A + (1-\lambda_2^2)\sum_{i=1}^{p}(AR_i + R_i^t A) + (\lambda_1^2+\lambda_2^2-2)\sum_{\substack{i,j=1\\ i\neq j}}^{p} R_i^t A R_j.$$

Therefore, we have

$$\begin{aligned} P_0(A) &= \frac{T^2(A) - (\lambda_1+\lambda_2)T(A) + \lambda_1\lambda_2 A}{(1-\lambda_1)(1-\lambda_2)} \\ &= \sum_{i=1}^{p}(AR_i + R_i^t A) - 2\sum_{\substack{i,j=1\\ i\neq j}}^{p} R_i^t A R_j. \end{aligned}$$

If (c) or (d) holds, then the expressions of $T(A)$ and $T^2(A)$ are

$$T(A) = \lambda_1^2 A + (\lambda_1 - \lambda_1^2)\sum_{i=1}^{p}(AR_i + R_i^t A) + (1-\lambda_1)^2(\sum_{\substack{i,j=1\\ i\neq j}}^{p} R_i^t A R_j,$$

$$T^2(A) = \lambda_1^4 A + (\lambda_1^2 - \lambda_1^4)\sum_{i=1}^{p}(AR_i + R_i^t A) + (1-\lambda_1^2)^2\sum_{\substack{i,j=1\\ i\neq j}}^{p} R_i^t A R_j.$$

Now, we have

$$P_0(A) = \sum_{\substack{i,j=1\\ i\neq j}}^{p} R_i^t A R_j.$$

This completes the proof of assertion (b) of Theorem 4.4. □

Proposition 4.10. *With the assumptions of Theorem 4.4, suppose that U has three distinct eigenvalues and only one with multiplicity greater than* 1*. Then $P_0(A)$ is equal to one of the following:*

(a) $AR_1 + R_1^t A - R_1^t AR_2 - R_2^t AR_1$;

(b) $A - (AR_1 + AR_2) + (AR_1 + AR_2)^t + 2(R_1^t AR_2 + R_2^t AR_1)$.

Proof. Suppose that U has three distinct eigenvalues, say, μ_1, μ_2, μ_3 with $n_1 > 1$. Thus, the spectrum of T will be

$$\{\mu_1^2, \mu_1\mu_2, \mu_2\mu_3, \mu_1\mu_3\} = \{1, \lambda_1, \lambda_2\}.$$

This is possible only if $\mu_1^2 = \mu_2\mu_3$. As a result, there are two possibilities:

(i) $\mu_1^2 = \lambda_1$, $\mu_1\mu_2 = 1$, $\mu_1\mu_3 = \lambda_2$ and

(ii) $\mu_1^2 = 1$, $\mu_1\mu_2 = \lambda_1$, $\mu_1\mu_3 = \lambda_2$.

If (i) holds, then $\lambda_1^2 = \mu_1^2\mu_1^2 = \mu_1^2\mu_2\mu_3 = \lambda_2$.

If (ii) holds, then $\lambda_1\lambda_2 = \mu_1\mu_2\mu_1\mu_3 = \mu_1^2\mu_2\mu_3 = \mu_1^4 = 1$.

Consequently the eigenvalues of U will have one of the following patterns:

(a) $\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_1}, \frac{1}{\sqrt{\lambda_1}}, \frac{\lambda_2}{\sqrt{\lambda_1}}$,

(b) $-\sqrt{\lambda_1}, \ldots, -\sqrt{\lambda_1}, -\frac{1}{\sqrt{\lambda_1}}, -\frac{\lambda_2}{\sqrt{\lambda_1}}$,

(c) $1, \ldots, 1, \lambda_1, \lambda_2$ or

(d) $-1, \ldots, -1, -\lambda_1, -\lambda_2$.

Thus, there exists a unitary matrix W such that

$$\begin{aligned} U &= W^* \operatorname{diag}(\mu_2, \mu_3, \mu_1, \dots, \mu_1) W \\ &= W^*[\mu_2 E_{11} + \mu_3 E_{22} + \mu_1(I - E_{11} - E_{22})]W \end{aligned}$$

Using the previous notation, we get

$$\begin{aligned} U &= \mu_2 R_1 + \mu_3 R_2 + \mu_1(I - R_1 - R_2) \\ &= \mu_1 I + (\mu_2 - \mu_1)R_1 + (\mu_3 - \mu_1)R_2 \end{aligned}$$

Now, we have $T(A) = U^t A U$

$$\begin{aligned} &= [\mu_1 A + (\mu_2 - \mu_1)R_1^t A + (\mu_3 - \mu_1)R_2^t A][\mu_1 I + (\mu_2 - \mu_1)R_1 + \\ &\quad (\mu_3 - \mu_1)R_2] = \mu_1^2 A + \mu_1(\mu_2 - \mu_1)(AR_1 + R_1^t A) + \mu_1(\mu_3 - \mu_1) \\ &\quad (AR_2 + R_2^t A) + (\mu_2 - \mu_1)(\mu_3 - \mu_1)(R_1^t A R_2 + R_2^t A R_1). \end{aligned}$$

Similarly, we have

$$\begin{aligned} T^2(A) &= \mu_1^4 A + \mu_1^2(\mu_2^2 - \mu_1^2)(AR_1 + R_1^t A) + \mu_1^2(\mu_3^2 - \mu_1^2)(AR_2 + R_2^t A) \\ &\quad + (\mu_2^2 - \mu_1^2)(\mu_3^2 - \mu_1^2)(R_1^t A R_2 + R_2^t A R_1). \end{aligned}$$

If (a) or (b) holds, then we have

$$\begin{aligned} T(A) &= \lambda_1 A + (1 - \lambda_1)(AR_1 + R_1^t A) + (\lambda_1^2 - \lambda_1)(AR_2 + R_2^t A) \\ &\quad - (1 - \lambda_1)^2 (R_1^t A R_2 + R_2^t A R_1), \end{aligned}$$

and

$$\begin{aligned} T^2(A) &= \lambda_1^2 A + (1 - \lambda_1^2)(AR_1 + R_1^t A) + (\lambda_1^4 - \lambda_1^2)(AR_2 + R_2^t A) \\ &\quad - (1 - \lambda_1^2)^2 (R_1^t A R_2 + R_2^t A R_1). \end{aligned}$$

Therefore, $P_0(A)$ will have the form

$$A \longmapsto AR_1 + R_1^t A - R_1^t A R_2 - R_2^t A R_1.$$

If (c) or (d) holds, then we have

$$\begin{aligned} T(A) &= A + (\lambda_1 - 1)(AR_1 + R_1^t A) + (\lambda_2 - 1)(AR_2 + R_2^t A) \\ &\quad + (2 - \lambda_1 - \lambda_2)(R_1^t A R_2 + R_2^t A R_1), \end{aligned}$$

and

$$\begin{aligned} T^2(A) &= A + (\lambda_1^2 - 1)(AR_1 + R_1^t A) + (\lambda_2^2 - 1)(AR_2 + R_2^t A) \\ &\quad + (2 - \lambda_1^2 - \lambda_2^2)(R_1^t A R_2 + R_2^t A R_1). \end{aligned}$$

Therefore, $P_0(A)$ will have the form

$$A \longmapsto A - (AR_1 + AR_2) + (AR_1 + AR_2)^t + 2(R_1^t A R_2 + R_2^t A R_1).$$

This completes the proof of assertion (c) of Theorem 4.4.

Hence, the proof of Theorem 4.4 is complete. □

5. Remarks

It is interesting to note here that the techniques used above to describe $G3P$s in the spaces of complex symmetric and skew-symmetric matrices may be used to describe GnPs as well, for $n > 3$. However, as it is evident from the proofs, the number of cases to be considered becomes increasingly larger and larger with greater values of n.

As pointed out in the Remark 4.5, the structure of $G3P$ on $K_n(\mathbb{C})$ is still unknown when $n = 4$. We end this paper by stating the following conjecture.

Conjecture 5.1. *Let $\|\cdot\|$ be a unitary congruence invariant norm on $K_4(\mathbb{C})$, and let P_0 be a $G3P$. Then the scalars associated with P_0 are cube roots of unity.*

Acknowledgments. I offer my gratitude to Dr. Sudipta Dutta for many insightful discussions throughout the preparation of this paper. I also would like to thank the referee for his/her careful review and valuable comments, which provided insights that helped to improve the paper. The referee made many suggestions that helped make the proofs precise and more elegant. The present versions of the proofs of Lemmas 4.6 and 4.7 are actually due to the referee.

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