REPRESENTATION OF GENERALIZED BI-CIRCULAR PROJECTIONS ON BANACH SPACES

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Abstract. We prove several results concerning the representation of projections on arbitrary Banach spaces. We also give illustrative examples including an example of a generalized bi-circular projection which can not be written as the average of the identity with an isometric reflection. We also characterize generalized bi-circular projections on $C_0(\Omega, X)$, with Ω a locally compact Hausdorff space (not necessarily connected) and X a Banach space with trivial centralizer.

1. Introduction

A projection P on a complex Banach space X is said to be a bi-circular projection if $e^{ia}P + e^{ib}(I - P)$ is an isometry, for all choices of real numbers a and b. These projections were first studied by Stacho and Zalar (in [13] and [14]) and shown to be norm hermitian by Jamison (in [11]).

Fošner, Iliševic and Li introduced a larger class of projections designated generalized bi-circular projections (henceforth GBP), cf. [6]. A generalized bi-circular projection P only requires that $P + \lambda(I - P)$ is an isometry, for some $\lambda \in \mathbb{T} \setminus \{1\}$. These projections are not necessarily norm hermitian. It is a consequence of the definition of a GBP that $P + \lambda(I - P)$ must be a surjective isometry, since

$$(P + \lambda(I - P))(y) = x, \text{ where } y = Px + \frac{1}{\lambda}(I - P)x, \ \forall\, x \in X.$$

In [6], the authors show that a generalized bi-circular projection on finite dimensional spaces is equal to the average of the identity with an isometric reflection. This interesting result was extended further to many other settings, as for example spaces of continuous functions on a compact, connected and Haursdorff space, $C(\Omega)$ and $C(\Omega, X)$, where generalized bi-circular projections are also represented

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as the average of the identity with an isometric reflection, see [4] and [10]. The same characterization also holds for generalized bi-circular projections on spaces of Lipschitz functions, see [5] and [16] and for L_p-spaces, $1 \leq p < \infty, p \neq 2$, see [12]. This raises the question whether every GBP on a Banach space is equal to the average of the identity with an isometric reflection. The answer to this question is negative as we show in example (2.6).

It is easy to see that there is a bijection between the set of all reflections on X and the set of all projections on X. If $P = \frac{Id+R}{2}$, with R an isometric reflection, is a GBP, then R is the identity on the range of P and $-I$ on the kernel of P.

In this note we show that given a GBP P on an arbitrary complex Banach space, P is hermitian or P is the average of the identity with a reflection R, with R an element in the algebra generated by the isometry associated with P. We give examples that show that the reflection defined by a GBP is not necessarily an isometry. Moreover, we also show that every projection on X is a GBP relative to some renorming of the underlying space X. Therefore in this new space, P can be represented as the average of the identity with an isometric reflection.

In section 3 we characterize projections written as combinations of iterates of a finite order operator and we relate those to the generalized n-circular projections discussed in [3] and also in [1]. In section 4 we derive the standard form for generalized bi-circular projections on $C_0(\Omega, X)$, with Ω a locally compact Hausdorff space (not necessarily connected) and X a Banach space with trivial centralizer.

2. A characterization of generalized bi-circular projection on a complex Banach space

Throughout this section X denotes a complex Banach space and P a bounded linear projection on X. We recall that P is a generalized bi-circular projection if and only if there exists a modulus 1 complex number $\lambda \neq 1$ such that $P + \lambda(I - P)$ is an isometry on X.

We observe that given an arbitrary projection P on X, $2P - I$ is a reflection and thus P can be represented as the average of the I with a reflection, i.e. $P = \frac{I+(2P-I)}{2}$. In particular, generalized bi-circular projections on X are averages of the identity with reflections. We recall that a reflection R on X is a bounded linear operator such that $R^2 = I$. An isometric reflection is both a reflection and an isometry. The next result represents the reflection determined by a GBP in terms of the surjective isometry defined by the projection.

Proposition 2.1. *Let X be a Banach space. If P is a projection such that $P + \lambda(I - P) = T$, where $\lambda \in \mathbb{T} \setminus \{1\}$ and T is an isometry on X, then $P = \frac{I+R}{2}$, with R, a reflection on X, in the algebra generated by T.*

Proof. Since λ is a modulus one complex number, it is of the form $e^{2\pi\theta i}$ with θ a real number in the interval $[0,1)$. Therefore, we consider the following two cases: (i) θ is an irrational number, and (ii) θ is a rational number. If θ is an irrational, then the sequence $\{\lambda^n\}_n$ is dense in the unit circle. This implies that P is a bi-circular projection since for every $\alpha \in \mathbb{T}$, $P+\alpha(I-P)$ is a surjective isometry, cf. [12]. If θ is a rational number, we first assume that λ is of even order. Thus for some positive integer k, $\lambda^k=-1$, $P+\lambda^k(I-P)=T^k$ and $P+\lambda^{2k}(I-P)=I=T^{2k}$. Consequently, P is represented as the average of the identity with the isometric reflection T^k. If λ is of odd order, let $2k+1$ be the smallest positive integer such that $\lambda^{2k+1}=1$. Therefore

$$P+\lambda^j(I-P)=T^j,\ \forall\ j=1,\dots 2k+1. \tag{1}$$

This implies that $T^{2k+1}=I$. Furthermore adding the equations displayed in (1), we get

$$(2k+1)P+(1+\lambda+\lambda^2+\cdots+\lambda^{2k})(I-P)=I+T+T^2+\ldots+T^{2k}.$$

This equation becomes

$$(2k+1)P=I+T+T^2+\ldots+T^{2k}, \tag{2}$$

since $1+\lambda+\lambda^2+\cdots+\lambda^{2k}=0$. The equation displayed in (2) implies that

$$P=\frac{1}{2k+1}\left(I+T+\cdots+T^{2k}\right)=\frac{I+R}{2},$$

with $R=\dfrac{(1-2k)I+2T+\cdots+2T^{2k}}{2k+1}$.

It is a straightforward calculation to check that $R^2=I$. This completes the proof.

□

Remark 2.2. *It follows from the proof given for the Theorem 2.1 that for θ irrational the projection P is bi-circular then hermitian. We now give an example that shows the converse of the implication in Theorem 2.1 does not hold. We consider X the space of all convergent sequences in $\mathbb{C}$ with the sup norm. Let $T : X \to X$ be given by $T(x_1,x_2,x_3,x_4,\cdots)=(x_2,x_3,x_1,x_4,\cdots)$, which involves a permutation of the first three positions of a sequence in X and the identity at any other position. It is clear that T is a surjective isometry and $P=\frac{I+T+T^2}{3}$ is a projection. As defined in the proof given for the Theorem 2.1, we set $R=\frac{-I+2T+2T^2}{3}$. The projection P is equal to $\frac{I+R}{2}$ and $R : X \to X$ is s.t. $R(x_1,x_2,x_3,x_4,\cdots)=\frac{1}{3}(-x_1+2x_2+2x_3,\ 2x_1-x_2+2x_3,\ 2x_1+2x_2-x_3, 3x_4,\ \cdots)$. Therefore, $R(0,1,1,0,\cdots)=\frac{1}{3}(4,1,1,0,0\cdots)$. This shows that R is not an isometry. It is easy to check that P is not a GBP. Given λ of modulus 1 and $\lambda\neq 1$, we* *set $S = (1-\lambda)P + \lambda I$. In particular,*

$$S(1,0,0,0,\cdots) = \left(\frac{1}{3} + \frac{2}{3}\lambda, \frac{1}{3} - \frac{1}{3}\lambda, \frac{1}{3} - \frac{1}{3}\lambda, 0, \cdots\right).$$

If S was an isometry on X, then $\max\{|\frac{1}{3} + \frac{2}{3}\lambda|, |\frac{1}{3} - \frac{1}{3}\lambda|\} = 1$. We observe that $|\frac{1}{3} - \frac{1}{3}\lambda| < 1$ and if $|\frac{1}{3} + \frac{2}{3}\lambda| = 1$, then $\lambda = 1$. This contradiction shows that P is not a GBP.

Given a projection it is of interest to determine whether P is a generalized bi-circular projection or equivalently whether the reflection determined by P is an isometry. We address this question in our next result.

Proposition 2.3. *Let X be a Banach space. If P is a projection on X such that $T = P + \lambda(I - P)$, for some $\lambda \in \mathbb{T} \setminus \{1\}$. Then, T is an isometry if and only if $\|x - y\| = \|x - \lambda y\|$, for every $x \in$ Range(P) and $y \in$ Ker(P).*

Proof. The projection P determines two closed subspaces Range(P) and Ker(P) such that $X = \text{Range}(P) \oplus \text{Ker}(P)$. Since T is an isometry, $\|x - y\| = \|Tx - Ty\|$ for every x and y in X. In particular for x in the range of P and y in the kernel of P, we have $Tx = x$ and $Ty = \lambda y$. The converse follows from straightforward computations.

$\square$

Remark 2.4. *If is a consequence of Proposition 2.3 that if P is a generalized bi-circular projection on X, then P is the average of the identity with an isometric reflection if and only if for every $x \in$ Range(P) and $y \in$ Ker(P), $\|x-y\| = \|x+y\|$.*

The next proposition asserts that every projection on a Banach space is a generalized bi-circular projection in some equivalent renorming of the given space.

Proposition 2.5. *Let X be a complex Banach space and P be a projection on X. Then X can be equivalently renormed such that R is an isometric reflection and consequently P is a generalized bi-circular projection.*

Proof. We set $R = 2P - I$. We observe that $R^2 = I$ which implies that R is bounded and bijective. Then, the Open Mapping Theorem implies that R is an isomorphism. Therefore, there exist α and β positive numbers such that, for every $x \in X$,

$$\alpha\|x\| \leq \|R(x)\| \leq \beta\|x\|.$$

We define $\|x\|_1 = \|x\| + \|R(x)\|$, for all $x \in X$. This new norm is equivalent to the original norm on X and R relative to this norm is an isometry. In fact, given $x \in X$, $\|R(x)\|_1 = \|R(x)\| + \|R(R(x))\| = \|x\|_1$.

$\square$

Example 2.6. *We now give an example of a GBP that can not be represented as the average of identity with an isometric reflection. Let X be $\mathbb{C}^3$ with the max norm, $\|(x,y,z)\|_\infty = \max\{|x|,|y|,|z|\}$ and $\lambda = exp(\frac{2\pi i}{3}) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. We consider P the following projection on $\mathbb{C}^3$:*

$$P(x,y,z) = \frac{1}{3}(x+y+z, x+y+z, x+y+z).$$

Let $T = P + \lambda(I-P)$. Straightforward computations imply that

$$T(x,y,z) = (a\,x + b\,(y+z), a\,y + b\,(x+z), a\,z + b\,(x+y)),$$

with $a = \frac{i\sqrt{3}}{3}$ and $b = \frac{1}{2} - \frac{\sqrt{3}i}{6}$.

Since $T(0,0,1) = (b,b,a)$, T is not an isometry. In fact, $\|(0,0,1)\|_\infty = 1$ and $\|T(0,0,1)\|_\infty = \frac{\sqrt{3}}{3} \neq \|(0,0,1)\|_\infty$. The isomorphism T has order 3 since $\lambda^3 = 1$.

We now renorm $\mathbb{C}^3$ so T becomes an isometry. The new norm is defined as follows:

$$\|(x,y,z)\|_* = \max\{\|(x,y,z)\|_\infty, \|T(x,y,z)\|_\infty, \|T^2(x,y,z)\|_\infty\}.$$

Therefore P is a generalized bi-circular projection in $\mathbb{C}^3$ with the norm $\|\cdot\|_*$, for $\lambda = exp(\frac{2\pi i}{3})$. This projection can not be written as the average of the identity with an isometric reflection. We assume otherwise, then $P = \frac{I+R}{2}$ and $R = \frac{-I+2T+2T^2}{3}$. We now show that R is not an isometry. Previous calculations imply that $T(0,0,1) = (b,b,a)$ and $T^2(0,0,1) = (b^2+2ab, b^2+2ab, a^2+2b^2) = (\overline{b},\overline{b},\overline{a})$. Therefore $R(0,0,1) = (2/3, 2/3, -1/3)$, $(TR)(0,0,1) = \frac{1}{3}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i, \frac{1}{2} + \frac{\sqrt{3}}{2}i, 2 - i\sqrt{3}\right)$, and $(T^2R)(0,0,1) = \frac{1}{3}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i, \frac{1}{2} - \frac{\sqrt{3}}{2}i, 2 + i\sqrt{3}\right)$.

Since $\|R(0,0,1)\|_\infty = \frac{2}{3}$, $\|(TR)(0,0,1)\|_\infty = \|(T^2R)(0,0,1)\|_\infty = \frac{\sqrt{7}}{3}$, we now conclude that

$$\|R(0,0,1)\|_* = \max\left\{\frac{2}{3}, \frac{\sqrt{7}}{3}\right\} = \frac{\sqrt{7}}{3} \neq \|(0,0,1)\|_* = 1.$$

It is worth mentioning that the projection P above does not satisfy the condition stated in Remark 2.4. For example, if $x = (1,1,1) \in Range(P)$, $y = (1,1,-2) \in Ker(P)$, we have $\|x+y\|_* = \sqrt{7}$ and $\|x-y\|_* = 3$.

3. Projections as combinations of finite order operators

In this section we investigate the existence of projections defined as linear combinations of the iterates of a given finite order operator. We conclude in our forthcoming Proposition 3.4 that only certain averages yield projections. For a generalized bi-circular projection P, we consider the set $\Lambda_P = \{\lambda \in \mathbb{T} : P + \lambda(I-P)$ is an isometry$\}$. This set is a group under multiplication. An inspection of the proof provided for the Theorem 2.1 also shows that the multiplicative group associated with a GBP is either finite or equal to $\mathbb{T}$. If Λ_P is infinite, then P is a bi-circular projection. We give some examples of GBPs together with their multiplicative groups.

Example 3.1. (1) *We consider ℓ_∞ with the usual sup norm. Let P be defined as follows:*

$$P(x_1, x_2, x_3, \cdots) = \left(\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2}, x_3, \cdots\right).$$

we show that $\Lambda_P = \{1, -1\}$. Given $\lambda \in \mathbb{T}$ such that $T = P + \lambda(I - P)$ is a surjective isometry, then

$$T(x_1, x_2, x_3, \cdots) = \left(\frac{(\lambda+1)x_1 + (1-\lambda)x_2}{2}, \frac{(\lambda+1)x_2 + (1-\lambda)x_1}{2}, x_3, \cdots\right).$$

We recall that a surjective isometry on ℓ_∞, $S : \ell_\infty \to \ell_\infty$ is of the form

$$S(x_1, x_2, x_3, \cdots) = (\mu_1 x_{\tau(1)}, \mu_2 x_{\tau(2)}, \cdots),$$

with τ a bijection of $\mathbb{N}$ and $\{\mu_i\}$ is a sequence of modulus 1 complex numbers.

Therefore T is an isometry if and only if $\lambda = \pm 1$.

(2) *Let P and T on $(\mathbb{C}^3, \|\cdot\|_*)$ be defined as in example (2.6). Then $\Lambda_P = \{1, exp(\frac{2\pi i}{3}), exp(\frac{4\pi i}{3})\}$. Since, $T = P + exp(\frac{2\pi i}{3})(I-P)$ is an isometry on $(\mathbb{C}^3, \|\cdot\|_*)$, then $T^2 = P + exp(\frac{4\pi i}{3})(I-P)$ is also an isometry and $\Lambda_P \supseteq \{1, exp(\frac{2\pi i}{3}), exp(\frac{4\pi i}{3})\}$. We now show that $\Lambda_P = \{1, exp(\frac{2\pi i}{3}), exp(\frac{4\pi i}{3})\}$. As in example (2.6), let $\lambda = exp(\frac{2\pi i}{3})$. Given $\lambda_0 = a_0 + ib_0$ of modulus 1, such that $\lambda_0 \notin \{1, exp(\frac{2\pi i}{3}), exp(\frac{4\pi i}{3})\}$, we set $S = P + \lambda_0(I-P)$. Therefore,*

$$S(x, y, z) = \frac{1}{3}(cx + d(y+z), cy + d(x+z), cz + d(x+y)),$$

with $c = 1 + 2\lambda_0$ and $d = 1 - \lambda_0$ and

$$\|S(0,0,1)\|_* = \max\{\|S(0,0,1)\|_\infty, \|TS(0,0,1)\|_\infty, \|T^2S(0,0,1)\|_\infty\}.$$

Now, $S(0,0,1) = \frac{1}{3}(d,d,c)$, $TS(0,0,1) = \frac{1}{3}(1-\lambda_0\lambda, 1-\lambda_0\lambda, 1+2\lambda_0\lambda)$ and $T^2S(0,0,1) = \frac{1}{3}(1-\lambda_0\lambda^2, 1-\lambda_0\lambda^2, 1+2\lambda_0\lambda^2)$. It is easy to see that each of $|\frac{1-\lambda_0}{3}|$, $|\frac{1-\lambda_0\lambda}{3}|$ and $|\frac{1-\lambda_0\lambda^2}{3}|$ is strictly less than 1. Moreover, if any of $|\frac{1+2\lambda_0}{3}|$, $|\frac{1+2\lambda_0\lambda}{3}|$ or $|\frac{1+2\lambda_0\lambda^2}{3}|$ is equal to 1, then $\lambda_0 = 1$, $\lambda_0 = \overline{\lambda}$ or $\lambda_0 = \overline{\lambda}^2$, respectively. This leads to a contradiction. It also follows from calculations already done for the example (2.6) that $\|(0,0,1)\|_* = 1$. Therefore, $\|S(0,0,1)\|_* \neq \|(0,0,1)\|_*$ and hence $\lambda_0 \notin \Lambda_P$.

The next corollary follows from our proof presented for Proposition 2.1.

Corollary 3.2. *Let X be a Banach space. If the order of the multiplicative group of a generalized bi-circular projection P on X is even then P is the average of the identity with an isometric reflection.*

We also recall the definition of a generalized n-circular projection, cf. [3]. A projection P on X is generalized n-circular if and only if there exists a surjective isometry T such that $T^n = I$ and

$$P = \frac{I + T + T^2 + \cdots + T^{n-1}}{n}.$$

Another notion of generalized n-circular projection was defined in [1] and it was shown there that both the definitions are equivalent in $C(\Omega)$, where Ω is a compact Hausdorff connected space. In fact, they are equivalent in any space in which the GBPs are given as the average of identity with an isometric reflection, see [2].

We observe that for a surjective linear map T on X such that $T^n = I$ (not necessarily an isometry), $\frac{I+T+T^2+\cdots+T^{n-1}}{n}$ is a projection. The same question applies to this situation; which spaces support only n-circular projections associated with surjective isometries?

We now show a result concerning existence of projections written as a linear combination of operators with a cyclic property.

Definition 3.3. *An operator T on X is of order k (a positive integer) if and only if $T^k = I$ and $T^i \neq I$ for any $i < k$.*

We observe that if T is of order k, then $P = \frac{I+T+T^2+\cdots+T^{k-1}}{n}$ is a projection. The following proposition answers the reverse question whether a combination of such a collection of operators yields any projection.

Before stating our result we set some useful notation as introduced in the book by Michael Frazier, [9]. We define $\rho = e^{-2\pi i/k}$. Then $\rho^{mn} = e^{-2\pi mni/k}$ and $\rho^{-mn} = e^{2\pi mni/k}$. In this notation, given a k-tuple $z = (z(0), \cdots, z(k-1))$ we set $\hat{z}(m) = \sum_{n=0}^{k-1} z(n)\rho^{mn}$. We now denote by W_k the k-square matrix with the (i,j) entry equal to $\rho^{(i-1)(j-1)}$. In expanded form

$$W_k = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \rho & \rho^2 & \rho^3 & \cdots & \rho^{k-1} \\ 1 & \rho^2 & \rho^4 & \rho^6 & \cdots & \rho^{2(k-1)} \\ 1 & \rho^3 & \rho^6 & \rho^9 & \cdots & \rho^{3(k-1)} \\ \vdots & \vdots & & & & \vdots \\ 1 & \rho^{k-1} & \rho^{2(k-1)} & \rho^{3(k-1)} & \cdots & \rho^{(k-1)(k-1)} \end{bmatrix}.$$

Regarding z and $\hat{z}$ as column vectors we have $\hat{z} = W_k z$. It is easy to see that W_k is invertible. The (i,j)-entry of W_k^{-1} is equal to $\frac{1}{k}\overline{\rho}^{(i-1)(j-1)}$. Frazier designates

$\hat{z}$ the "discrete Fourier transform" of z, i.e., $\hat{z} = DFT(z)$, and z is the "inverse discrete Fourier transform" of $\hat{z}$, i.e., $z = W_k^{-1}\hat{z} = IDFT(\hat{z})$. If S is a subset of $\{0, \cdots, k-1\}$, we denote by δ_S the vector with components given by $\delta(i) = 1$ for $i \in S$ and $\delta(i) = 0$ otherwise.

Proposition 3.4. *Let X be a Banach space and P a bounded operator on X. Let $\lambda_0, \cdots, \lambda_{k-1}$ be nonzero complex numbers and $P = \sum_{i=0}^{k-1} \lambda_i T^i$, where T is an operator of order k. Then, P is a projection if and only if $\lambda = (\lambda_0, \lambda_1, \cdots, \lambda_{k-1})$ is the IDFT of δ_S, for some $S \subseteq \{0, \cdots, k-1\}$.*

Proof. If $P = \sum_{i=0}^{k-1} \lambda_i T^i$ and T is an algebraic operator with annihilating polynomial $x^k - 1$, a Theorem due to Taylor (cf. [15] p. 317-318) asserts that

$$T = Q_0 + \rho Q_1 + \cdots + \rho^{k-1} Q_{k-1}$$

with $\{Q_0, \cdots, Q_{k-1}\}$ pairwise orthogonal projections. Since $T^i = Q_0 + \rho^i Q_1 + \cdots + \rho^{i(k-1)} Q_{k-1}$ we conclude that $P = \alpha_0 Q_0 + \alpha_1 Q_1 + \cdots + \alpha_{k-1} Q_{k-1}$ with the vector of scalars $(\alpha_0, \cdots, \alpha_{k-1})$ equal to the $DFT(\lambda_0, \lambda_1, \cdots, \lambda_{k-1})$. Since P is a projection, i.e., $P^2 = P$ and $\{Q_0, \cdots, Q_{k-1}\}$ are pairwise orthogonal projections we have that $\alpha_i^2 = \alpha_i$, for $i = 0, \cdots, k-1$. On the other hand,

$$P = \sum_{i=0}^{k-1} \lambda_i T^i = \sum_{i=0}^{k-1} \left(\sum_{j=0}^{k-1} \lambda_j \rho^{ij} \right) Q_i,$$

thus for $i = 0, \cdots, k-1$, $\alpha_i = \sum_{j=0}^{k-1} \lambda_j \rho^{ij}$. This implies that $(\lambda_0, \lambda_1, \cdots, \lambda_{k-1})$ is the $IDFT(\delta_S)$ for some S a subset of $\{0, \cdots, k-1\}$.

Conversely, we associate with T the collection $Q_0, \cdots, Q_{k-1}$ of k pairwise orthogonal projections, such that the range of each Q_i is the eigenspace associated with the eigenvalue ρ^i. Then $\delta_S(0)Q_0 + \delta_S(1)Q_1 + \cdots + \delta_S(k-1)Q_{k-1} = \sum_{i=0}^{k-1} \lambda_i T^i$, and $P = \delta_S(0)Q_0 + \delta_S(1)Q_1 + \cdots + \delta_S(k-1)Q_{k-1}$ is clearly a projection. This completes the proof.

□

4. Spaces of Vector-Valued Functions

In this section we characterize generalized bi-circular projections on spaces of continuous functions defined on a locally compact Hausdorff space. This characterization extends results presented in [3] and [4] for compact and connected Hausdorff spaces. We recall a folklore lemma which is very easy to prove.

Lemma 4.1. *Let X be a Banach space and $\lambda \in \mathbb{T} \setminus \{1\}$. Then the following are equivalent.*

(a) *T is a bounded operator on X satisfying $T^2 - (\lambda + 1)T + \lambda I = 0$.*

(*b*) *There exists a projection* P *on* X *such that* $P + \lambda(I - P) = T$.

Theorem 4.2. *Let* Ω *be a locally compact Hausdorff space, not necessarily connected, and* X *be a Banach space which has trivial centralizer. Let* P *be a GBP on* $C_0(\Omega, X)$. *Then one and only one of the following holds.*

(*a*) $P = \frac{I+T}{2}$, *where some* T *is an isometric reflection on* $C_0(\Omega, X)$.
(*b*) $Pf(\omega) = P_\omega(f(\omega))$, *where* P_ω *is a generalized bi-circular projection on* X.

Proof. Let $P + \lambda(I - P) = T$, where $\lambda \in \mathbb{T} \setminus \{1\}$ and T is an isometry on $C_0(\Omega, X)$. From [8], we see that T has the form $Tf(\omega) = u_\omega(f(\phi(\omega)))$ for $\omega \in \Omega$ and $f \in C_0(\Omega, X)$, where $u : \Omega \to \mathcal{G}(X)$ continuous in strong operator topology and ϕ is a homeomorphism of Ω onto itself. Here, $\mathcal{G}(X)$ is the group of all surjective isometries on X. From Lemma 4.1, we have $T^2 - (\lambda + 1)T + \lambda I = 0$. That is

$$u_\omega \circ u_{\phi(\omega)}(f(\phi^2(\omega))) + (\lambda + 1)u_\omega(f(\phi(\omega))) + \lambda f(\omega) = 0. \tag{3}$$

Let $\omega \in \Omega$. If $\phi(\omega) \neq \omega$, then $\phi^2(\omega) = \omega$. For otherwise, there exists $h \in C_0(\Omega)$ such that $h(\omega) = 1$, $h(\phi(\omega)) = h(\phi^2(\omega)) = 0$. For $f = h \bigotimes x$, where x is a fixed vector in X, Equation (3) reduces to $\lambda = 0$, contradicting the assumption on λ. Now, choosing $h \in C_0(\Omega)$ such that $h(\omega) = 0$, $h(\phi(\omega)) = 1$ we get $\lambda = -1$. This implies that $u_\omega \circ u_{\phi(\omega)} = I$. If $\phi(\omega) = \omega$ and ϕ is not the identity, then since we will have the above case (i.e., $\phi(\omega) \neq \omega$) for some $\omega' s$, we conclude that $\lambda = -1$. This again implies that $u_\omega^2 = I$. Hence in both cases P will be of the form $\frac{I+T}{2}$ and $T^2 = I$.

If $\phi(\omega) = \omega$ for all $\omega \in \Omega$, then we will have from Equation (2)

$$u_\omega^2 - (\lambda + 1)u_\omega + \lambda I = 0.$$

Thus from Lemma 4.1, there exists a projection P_ω on X such that $P_\omega + \lambda(I - P_\omega) = u_\omega$. Since u_ω is an isometry, P_ω is a GBP. Therefore, we have $Pf(\omega) = P_\omega(f(\omega)$. This completes the proof.

□

Corollary 4.3. *Let* Ω *be a locally compact Hausdorff space (not necessarily connected) and* P *be a GBP on* $C_0(\Omega)$. *Then one and only one of the following holds.*

(*a*) $P = \frac{I+T}{2}$, *where* T *is an isometric reflection on* $C_0(\Omega)$.
(*b*) P *is a bi-circular projection.*

Remark 4.4. *Similar results were proved in* [4] *for* $C(\Omega, X)$, *with* Ω *connected. Here we extend those results to more general settings.*

It was proved in [7] that if (X_n) is a sequence of Banach spaces such that every X_n has trivial L_∞ structure, then any surjective isometry of $\bigoplus_{c_0} X_n$ is of the form

$(Tx)_n = U_{n\pi(n)} x_{\pi(n)}$ for each $x = (x_n) \in \bigoplus_{c_0} X_n$. Here π is a permutation of $\mathbb{N}$ and $U_{n\pi(n)}$ is a sequence of isometric operators which maps $X_{\pi(n)}$ onto X_n.

Suppose P is a GBP on $\bigoplus_{c_0} X_n$, then similar techniques employed in the proof of Theorem 4.2 also prove the following result.

Theorem 4.5. *Let P is a a generalized bi-circular projection on $\bigoplus_{c_0} X_n$. Then one and only one of the following holds.*

(*a*) $P = \frac{I+T}{2}$, *where T is an isometric reflection on $\bigoplus_{c_0} X_n$.*

(*b*) $(Px)_n = P_n x_n$ *where P_n is a generalized bi-circular projection on X_n.*

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