Indian Institute of Information Technology Allahabad
Univariate and Multivariate Calculus
C3 End Semester Examination - Tentative Marking Scheme

Program: B.Tech. 2^{nd} Semester
Duration: **2 hours** Full Marks: 55
Date: June 19, 2023 Time: 4:30 PM - 06:30 PM

Important Instructions:

1. Answer all questions. Writing on question paper is not allowed.
2. Attempt all the parts of questions 2 at the same place. Parts done separately will not be graded.
3. Number the pages of your answer booklet. On the back of the front page of your answer booklet, make a table (as shown below) to indicate the page number in which respective questions have been answered. If you did not attempt a particular question, write down NA.

Question No.	1	2	3	4	5	6	7	8
Page No.								

4. Use of any electronic gadgets is not allowed.
5. This question paper contains two pages.

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1. Let $f : [0,1] \to \mathbb{R}$ be a function defined as

$$f(x) = \begin{cases} \frac{1}{2^n} & \frac{1}{2^{n+1}} < x \leq \frac{1}{2^n} \\ 0 & x = 0, \end{cases}$$

where $n = 0, 1, 2, \ldots$. Show that f is integrable and find its integral value. [6]

Solution. It is clear that f is bounded and increasing on $[0,1]$. Hence f is integrable. [1]

By the fundamental theorem of calculus, the function $F : [0,1] \to \mathbb{R}$ defined by

$$F(x) = \int_x^1 f(t)dt$$

is continuous. So, if $x_n \to 0$, then $F(x_n) \to F(0)$ that is,

$$F(0) = \int_0^1 f(t)dt = \lim_{n\to\infty} \int_{x_n}^1 f(t)dt. \qquad [2]$$

Consider $x_n = \frac{1}{2^n}$. Then

$$F\left(\frac{1}{2^n}\right) = \int_{\frac{1}{2^n}}^{1} f(t)dt$$
$$= \int_{\frac{1}{2^n}}^{\frac{1}{2^{n-1}}} f(t)dt + \int_{\frac{1}{2^{n-1}}}^{\frac{1}{2^{n-2}}} f(t)dt + \cdots + \int_{\frac{1}{2}}^{1} f(t)dt$$
$$= \frac{1}{2}\left\{1 + \frac{1}{2^2} + \cdots + \left(\frac{1}{2^2}\right)^{n-1}\right\}$$
$$= \frac{2}{3}\left(1 - \frac{1}{4^n}\right) \qquad [2]$$

Hence,

$$\int_0^1 f(t)dt = \lim_{n\to\infty} \int_{\frac{1}{2^n}}^{1} f(t)dt = \frac{2}{3}. \qquad [1]$$

2. Consider the function

$$f(x,y) = \begin{cases} \frac{3x^2y - y^3}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

Answer the following questions.

(a) Discuss the continuity of f at $(0,0)$. [2]

Solution.

$$|f(x,y) - f(0,0)| = \left|\frac{y(3x^2 - y^2)}{x^2+y^2}\right| \leq \left|\frac{y(3x^2 + 3y^2)}{x^2+y^2}\right| = |3y| \to 0 \text{ as } (x,y) \to 0.$$

Thus, f is continuous at $(0,0)$. [2]

(b) Is f_y continuous at $(0,0)$. [3]

Solution. $f_y(x,0) = \lim_{h\to 0} \frac{f(x,h) - f(x,0)}{h} = \lim_{h\to 0} \frac{3x^2 - h^2}{x^2 + h^2} = 3.$ [1]

$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = -1.$ [1]

Since $f_y(x,0) \nrightarrow f_y(0,0)$ as $x \to 0$, f_y is not continuous at $(0,0)$. [1]

(c) Find the directional derivative of f at $(0,0)$ in the direction of $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. [2]

Solution. The directional derivative of f at $(0,0)$ in the direction of $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is

$$D_{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)} f(0,0) = \lim_{t\to 0} \frac{f((0,0) + t\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)) - f(0,0)}{t} \qquad [1]$$
$$= \lim_{t\to 0} \frac{\frac{t^3}{\sqrt{2}}}{t^3} = \frac{1}{\sqrt{2}} \qquad [1]$$

(d) Discuss the differentiability of f at $(0,0)$. [3]

Solution. $f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = 0.$ [1]

Let $H = (h,k) \in \mathbb{R}^2$. Then the error function is

$$\epsilon(H) = \frac{f(h,k) - f(0,0) - hf_x(0,0) - kf_y(0,0)}{||H||} = \frac{4h^2k}{(h^2+k^2)^{\frac{3}{2}}}. \quad [1]$$

Take $h = k$. Then $\epsilon(h,h) = \sqrt{2} \nrightarrow 0$ as $h \to 0$. Therefore, f is not differentiable at $(0,0)$. [1]

Alternative Solution. If f is differentiable at (a,b), then the directional derivative of f at (a,b) in the direction of a unit vector (u_1,u_2) is
$D_{(u_1,u_2)}f(a,b) = (f_x(a,b), f_y(a,b)).U.$

Now, $f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = 0.$ [1]

Since

$$D_{\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)}f(0,0) = \frac{1}{\sqrt{2}},$$

but

$$(f_x(0,0), f_y(0,0)).\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = (0,-1).\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}}, \quad [2]$$

f is not differentiable at $(0,0)$.

3. Let

$$f(x,y) = x^2 + 2bxy + y^2,\ b \in \mathbb{R}.$$

Find all the critical points and classify them as point of local maxima, local minima or saddle point. [10]

Solution. The partial derivatives of f w.r.t. x and y are given by

$$f_x(x,y) = 2x + 2by \text{ and } f_y(x,y) = 2bx + 2y$$

respectively.

The critical points are $(0,0)$, [1]

(x,x), $x \in \mathbb{R}$ when $b = -1$ and [1]

$(x,-x)$, $x \in \mathbb{R}$ when $b = 1$. [1]

Now, $f_{xx}(x,y) = 2$, $f_{yy}(x,y) = 2$, $f_{xy}(x,y) = 2b$.

The Discriminant of f at (x,y) is $\Delta f(x,y) = 4 - 4b^2$. [1]

At $(0,0)$, $f_{xx}(0,0) > 0$, $\Delta f(0,0) > 0$ if and only if $b^2 < 1$.

Thus, if $b^2 < 1$, $(0,0)$ is a point of minima. [1]

If $b^2 > 1$, $(0,0)$ is a saddle point. [1]

If $b^2 = 1$, (x,x) and $(x,-x)$ are critical points and $\Delta(x,\pm x) = 0$, then the test is inconclusive.

If $b^2 = 1$, then $f(x,\pm x) = 0 \le f(x,y) = (x-y)^2$, $\forall\ x,y \in \mathbb{R}$. [2]

This implies that (x,x) and $(x,-x)$ are points of local minima. [2]

4. The region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ is revolved about the y-axis to generate a solid. Find the volume of the solid using Shell method. [6]

Solution. Using Shell method, the volume of the solid generated is given by

$$V = \int_a^b 2\pi \text{ (shell radius)(shell height)} dx. \qquad [1]$$

Shell Radius $= x$, Shell Height $= \sqrt{x}$. [1+1]

Thus,

$$V = \int_0^4 2\pi x(\sqrt{x})dx = 2\pi \int_0^4 x^{3/2} dx = \frac{128\pi}{5}. \qquad [1+1+1]$$

5. Find the length of the curve $r = 1 - \cos\theta$. [4]

Solution. The length of the curve in polar coordinates is given by

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta. \qquad [1]$$

It follows that

$$L = \int_0^{2\pi} \sqrt{2 - 2\cos\theta} d\theta = 8. \qquad [1+1+1]$$

6. The curve

$$x = \cos t, \quad y = 1 + \sin t, \quad 0 \le t \le 2\pi.$$

is revolved about the x-axis. Using this parametrization, find the area of the surface generated. [4]

Solution. The surface area is given by

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt. \qquad [1]$$

This implies that

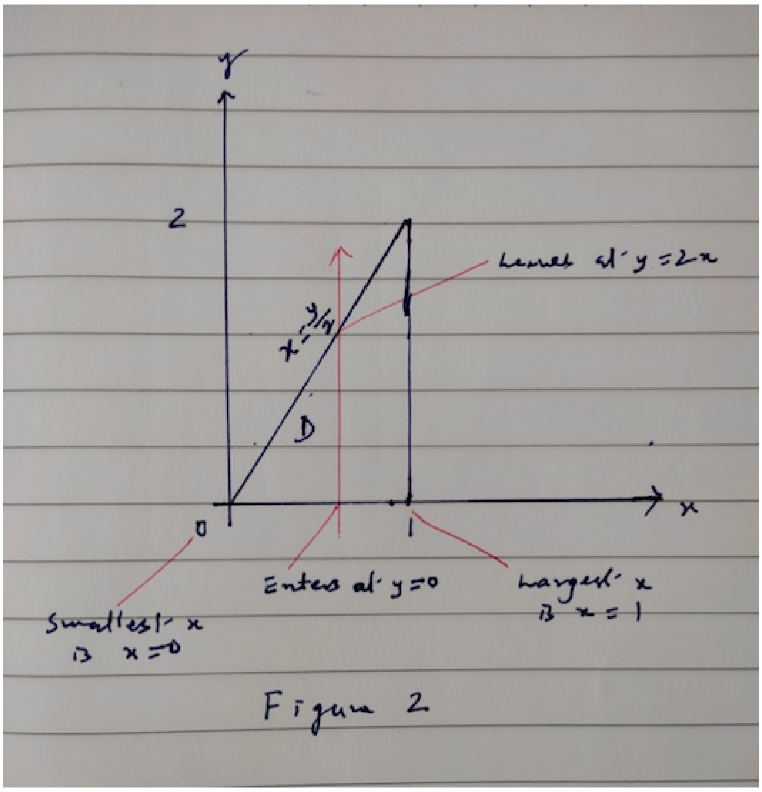
$$S = \int_0^{2\pi} 2\pi(1 + \sin t)\sqrt{(-\sin t)^2 + (\cos t)^2} dt = 2\pi \int_0^{2\pi} (1 + \sin t) dt = 4\pi^2. \qquad [1+1+1]$$

7. Evaluate the integral [5]

$$\int_0^2 \left(\int_{\frac{y}{2}}^1 e^{x^2} dx \right) dy.$$

Solution. The region D of integration (see Figure 2) is given by $D = \{(x, y) \in \mathbb{R}^2 : 0 \le y \le 2, \frac{y}{2} \le x \le 1\}$. Since the integral $\int_{y/2}^{1} e^{x^2} dx$ cannot be evaluated in terms of simple known functions, we will use Fubini's theorem and change the order of integration. The region of integration becomes $D = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq x \leq 2x\}$ (see Figure 2). Thus,

$$\iint_D f(x, y) dy dx = \int_0^1 \left(\int_0^{2x} e^{x^2} dy \right) dx = e - 1. \qquad [1+1+1+1+1]$$



8. Let $D = \{(x, y, z) : x^2 + y^2 + z^2 \leq 4a^2, z \geq a\}$. Evaluate the integral

$$\iiint_D \frac{z}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} dx dy dz,$$

using spherical coordinates. [10]

Solution. If we allow ϕ to vary independently, then ϕ varies from 0 to $\frac{\pi}{3}$ (see Figure 2). [1+1]

For fixed ϕ and ρ, θ to varies from 0 to 2π. [1+1]

For a fixed ϕ and θ, ρ varies from $a \sec \phi$ to $2a$ (see Figure 1). [1+1]

Therefore, the integral is

$$\int_0^{\frac{\pi}{3}} \int_0^{2\pi} \int_{a \sec \phi}^{2a} f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) |J(\rho, \theta, \phi)| d\rho d\theta d\phi$$

$$= \int_0^{\frac{\pi}{3}} \int_0^{2\pi} \int_{a \sec \phi}^{2a} \frac{cos\phi}{\rho^2} \rho^2 \sin \phi d\rho d\theta d\phi \qquad [1+1]$$

$$= 2\pi \int_0^{\frac{\pi}{3}} (2a \sin \phi \cos \phi - a \sin \phi) d\phi = \frac{\pi a}{2}. \qquad [2]$$

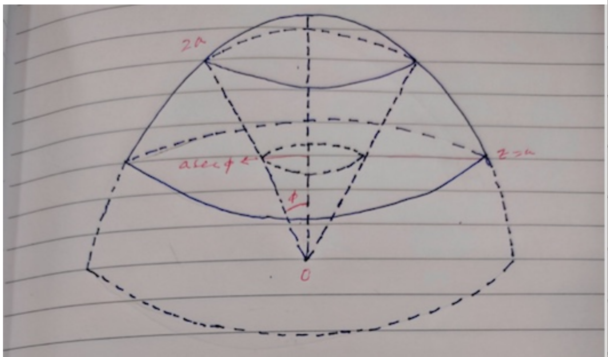


Figure 1

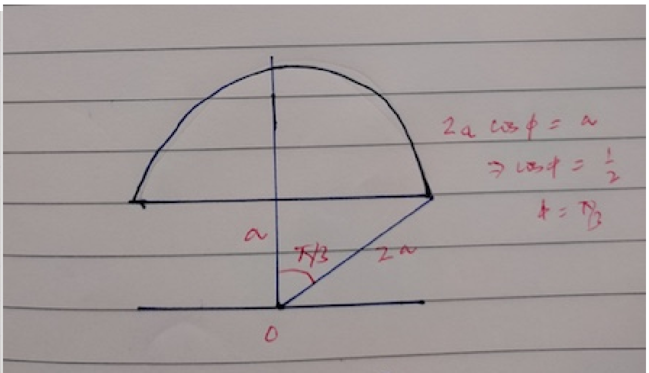


Figure 2