Indian Institute of Information Technology Allahabad
Univariate and Multivariate Calculus
C2 Quiz

Program: B.Tech. 2^{nd} Semester
Duration: **45 minutes** Full Marks: 20
Date: May 24, 2023 Time: 09:00 AM - 09:45 AM

Attempt all questions.

1. Sketch the graph of the function [12]

$$f(x) = \frac{x^2+4}{2x}$$

after finding the intervals of decrease/increase, intervals of concavity/convexity, points of local minima/local maxima, points of inflection, and asymptotes.

Solution: $f'(x) = \frac{1}{2} - \frac{2}{x^2} = \frac{x^2-4}{2x^2}$, $f''(x) = \frac{4}{x^3}$.

The critical points occur at $x = 2, -2$ where $f'(x) = 0$.

Since $f''(-2) < 0$ and $f''(2) > 0$, we see from the second derivative test that a relative maximum occurs at $x = -2$ with $f(-2) = -2$, and a relative minimum occurs at $x = 2$ with $f(2) = 2$. [1+1]

On the interval $(-\infty, -2)$,the derivative f' is positive because $x^2 - 4 > 0$ so the graph is increasing; on the interval $(-2, 0)$ the derivative is negative and the graph is decreasing. Similarly, the graph is decreasing on $(0, 2)$ and increasing on $(2, \infty)$. [1+1+1+1]

The graph is concave on the interval $(-\infty, 0)$ as $f''(x) < 0$ whenever $x < 0$ and convex on the interval $(0, \infty)$ as $f''(x) > 0$ whenever $x > 0$. [1+1]

Since 0 is not in the domain, there is no points of inflection. [1]

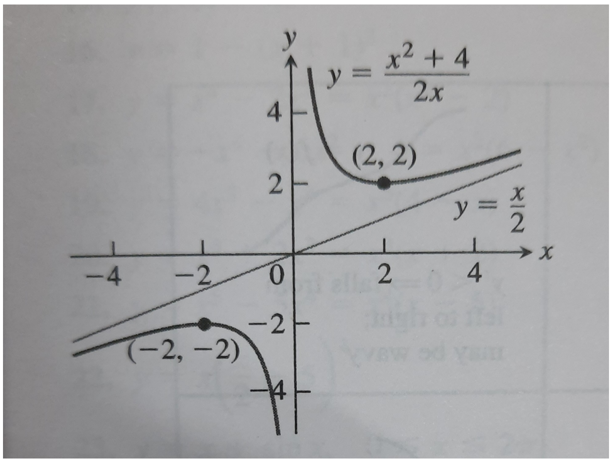
Now, $f(x) = \frac{x^2+4}{2x} = \frac{x}{2} + \frac{2}{x}$, $lim_{x\to 0^+} \left(\frac{x}{2} + \frac{2}{x}\right) = +\infty$ and $lim_{x\to 0^-} \left(\frac{x}{2} + \frac{2}{x}\right) = -\infty$.
so the y- axis is a vertical asymptote. [1]

Also, as $x \to \infty$ or $x \to -\infty$, the graph of $f(x)$ approaches the line $y = \frac{x}{2}$. Thus, $y = \frac{x}{2}$ is an oblique asymptote. [1].

2. Let $f : [a, b] \to \mathbb{R}$ be a function and n be a non-negative integer. Suppose $f^{(n+1)}$ exists and is identically zero on $[a, b]$. Show that f is a polynomial function of degree less than or equal to n. [2]

Proof. Assume that $f : [a, b] \to \mathbb{R}$ is such that $f^{(n)}$ is continuous on $[a, b]$ and $f^{(n+1)}(x)$ exists on (a, b). Fix $x_0 \in [a, b]$. Then Taylor's theorem says, for each $x \in [a, b]$ with $x \neq x_0$, there exists c between x and x_0 such that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$



[1]

Figure 1: Graph for Question 1

Take any $x \in (a, b]$. Apply Taylor's theorem for f on $[a, x]$ and use $f^{(n+1)}(c) = 0$, we get that

$$f(x) = f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

which is a polynomial of degree $\leq n$. □

3. Discuss the convergence/divergence of the following series: [3+3]

   (a) $\sum_{n=1}^{\infty} e^{-n^2}$.

   (b) $\sum_{n=1}^{\infty} \sin\left(\frac{(-1)^n}{n^p}\right)$, $p > 0$.

Solution:

(a) Observe that $\frac{1}{e^{n^2}} \leq \frac{1}{n^2}$, for all $n \in \mathbb{N}$. [1]

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent series, by comparison test, $\sum_{n=1}^{\infty} e^{-n^2}$ is also convergent. [1+1]

(b) $\sin\left(\frac{(-1)^n}{n^p}\right) = (-1)^n \sin\left(\frac{1}{n^p}\right)$. [1]

Also, $\sin\left(\frac{1}{n^p}\right)$ is a decreasing sequence. [1]

Since $\sin\left(\frac{1}{n^p}\right)$ coverges to 0, by Leibnitz test, the series is convergent. [1]