Indian Institute of Information Technology Allahabad
Univariate and Multivariate Calculus
C1 Review Test Tentative Marking Scheme

Program: B.Tech. 2^{nd} Semester
Duration: **1 hour 15 minutes** Full Marks: 35
Date: May 03, 2023 Time: 6:15 PM - 07:30 PM

Important Instructions:

1. Answer all questions. Writing on question paper is not allowed.
2. Attempt all the parts of questions 1 at the same place. Parts done separately will not be graded.
3. Number the pages of your answer booklet. On the back of the front page of your answer booklet, make a table (as shown below) to indicate the page number in which respective questions have been answered. If you did not attempt a particular question, write down NA.

Question No.	1	2	3	4	5	6	7
Page No.							

4. Use of any electronic gadgets is not allowed.

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1. Discuss the convergence/divergence of the following sequences, and find the limit. [12]

   (a) $x_n = (\sqrt{2} - 2^{\frac{1}{3}})(\sqrt{2} - 2^{\frac{1}{5}}) \cdots (\sqrt{2} - 2^{\frac{1}{2n+1}})$.
   Solution. We observe that for $n \in \mathbb{N}$, $2^{\frac{1}{2n+1}} > 1$, which implies that $\sqrt{2} - 2^{\frac{1}{2n+1}} < \sqrt{2} - 1$. [2]
   Thus, $0 < x_n < (\sqrt{2} - 1)^n$. By Sandwich theorem, $x_n \to 0$. [1]

   (b) $x_n = (n!)^{1/n}$.
   Solution. Let $r > 0$ be given. Since $\frac{r^n}{n!} \to 0$, for $\varepsilon = 1$, there exists $N \in \mathbb{N}$ such that $\frac{r^n}{n!} < 1$ whenever $n \geq N$. [2]
   That is, $r^n < n!$ or $(n!)^{1/n} > r$ for $n \geq N$. Therefore, $(n!)^{1/n} \to \infty$. [2]

   (c) $x_1 = 1$ and $x_{n+1} = \frac{4+3x_n}{3+2x_n}$.
   Solution. Note that $x_2 > x_1$. Since $x_{n+1} - x_n = \frac{x_n - x_{n-1}}{(3+2x_n)(3+2x_{n-1})}$, by induction (x_n) is increasing. [2]
   Moreover, $x_{n+1} = 1 + \frac{1+x_n}{3+2x_n} \leq 2$. Thus, (x_n) is bounded above, and hence convergent. [2]
   The limit is $\sqrt{2}$. [1]

2. Let $f, g : [a, b] \to \mathbb{R}$ be two functions such that $g(x) > 0$ for all $x \in [a, b]$, $x \neq c$, $\lim_{x\to c} f(x) > 0$ and $\lim_{x\to c} g(x) = 0$. Prove that $\lim_{x\to c} \frac{f(x)}{g(x)} = \infty$. (Hint: Prove using $\epsilon - \delta$ definition of limits) [6]

Solution. Let $r > 0$ and $\lim_{x\to c} f(x) = A > 0$. There exists $\delta_1, \delta_2 > 0$, such that

$$0 < |x - c| < \delta_1 \Longrightarrow |f(x) - A| < A/2 \text{ or } f(x) > A/2, \qquad [2]$$

and

$$0 < |x - c| < \delta_2 \Longrightarrow g(x) < A/2r \text{ or } \frac{1}{g(x)} > 2r/A. \qquad [2]$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then

$$0 < |x - c| < \delta \Longrightarrow \frac{f(x)}{g(x)} > r. \qquad [2]$$

Therefore, $\lim_{x\to c} \frac{f(x)}{g(x)} = \infty$.

3. Does there exist a continuous function from $[0,1]$ onto $(0,1)$. Justify your answer. [2]

   Solution. No. If f is a continuous function from $[0,1]$ onto $(0,1)$, then f is bounded and attains its bounds, i.e., there exists $c \in [0,1]$ such that $f(c) = \inf\{f(x) : x \in [0,1]\} = \inf(0,1) = 0$. This is impossible. [2]

4. Suppose $f : [0,\infty) \to \mathbb{R}$ is continuous and $\lim_{x\to\infty} f(x)$ exists. Show that f is bounded. [5]

   Solution. Suppose $\lim_{x\to\infty} f(x) = \ell$. Then there exists $r \in \mathbb{R}$ such that $|f(x) - \ell| < 1$ for all $x \geq r$. [2]

   Then $|f(x)| \leq 1 + \ell$ for all $x \geq r$, i.e., f is bounded on $\{x : x \geq r\}$. [1]

   Since f is continuous, it is bounded on $[0, r]$. [2]

   Therefore, f is bounded on $[0, \infty]$.

5. Find the number of real solutions of the equation [4]

$$x^{17} - e^{-x} + 5x - \cos x = 0.$$

   Solution. Since $f(2) > 0$ and $f(-2) < 0$, by IVP f has at least one solution. [2]

   Since $f'(x) > 0$ for all $x \in \mathbb{R}$, Rolle's theorem implies that f has exactly one solution. [2]

6. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable at $x = 1$, $f(1) = 1$ and $k \in \mathbb{N}$. Show that

$$\lim_{n\to\infty} n\left[f\left(1 + \frac{1}{n}\right) + f\left(1 + \frac{2}{n}\right) + \cdots + f\left(1 + \frac{k}{n}\right) - k\right] = \frac{k(k+1)}{2} f'(1). \qquad [4]$$

   Solution. $\lim_{n\to\infty} n\left[f\left(1 + \frac{1}{n}\right) + f\left(1 + \frac{2}{n}\right) + \cdots + f\left(1 + \frac{k}{n}\right) - k\right]$

$= \left[\lim_{n\to\infty} \frac{f\left(1+\frac{1}{n}\right)-1}{\frac{1}{n}}\right] + 2\left[\lim_{n\to\infty} \frac{f\left(1+\frac{2}{n}\right)-1}{\frac{2}{n}}\right] + \cdots + k\left[\lim_{n\to\infty} \frac{f\left(1+\frac{k}{n}\right)-1}{\frac{k}{n}}\right]$ [2]

$= \left[\lim_{n\to\infty} \frac{f\left(1+\frac{1}{n}\right)-f(1)}{\frac{1}{n}}\right] + 2\left[\lim_{n\to\infty} \frac{f\left(1+\frac{2}{n}\right)-f(1)}{\frac{1}{n}}\right] + \cdots + k\left[\lim_{n\to\infty} \frac{f\left(1+\frac{k}{n}\right)-f(1)}{\frac{1}{n}}\right]$

$= (1 + 2 + \cdots + k) f'(1)$ [2]

$= \frac{k(k+1)}{2} f'(1)$.

7. Let $f : [a,b] \to \mathbb{R}$ be differentiable such that $f'(x) = 0$ for all $x \in [a,b]$. Show that f is constant. [2]

   Solution. For any $x, y \in [a,b]$, by MVT, we have $f(y) - f(x) = f'(z)(y - x)$ for some z between x and y. Since $f'(z) = 0$, we get $f(x) = f(y)$ for all $x, y \in [a,b]$. This implies that f is a constant function. [2]