Indian Institute of Information Technology Allahabad
End Sem Question Paper
Marking Scheme

Important Instructions:

1. All notations are standard, and the same as used in classes and lecture notes.

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1. Check whether the following statements are true or false. Give a proper justification. [8]

(a) Let $\sigma = (153)(264)$ be a permutation in S_7. Then σ is an even permutation.
Solution: True.
Since $\sigma = (153)(264) = (13)(15)(24)(26)$, σ is even permutation.

(b) Let V be a vector space of dimension n. Let W_1 be a subspace of dimension $n-1$ and W_2 be a subspace of dimension r such that W_2 is not contained in W_1. Then the dimension of $W_1 \cap W_2$ is $n-2$.
Solution: False.
$\min\{\dim(W_1), \dim(W_2)\} \geq \dim(W_1 \cap W_2) \geq \dim(W_1) + \dim(W_2) - dim(V)$.
Since W_1 and W_2 are distinct, $r > \dim(W_1 \cap W_2) \geq r-1$.
So $\dim(W_1 \cap W_2) = r-1$.

(c) There exists a proper subspace ($\neq \mathbb{R}$) of $\mathbb{C}(\mathbb{R})$ containing $\mathbb{R}$.
Solution: False.
We know that $\dim(\mathbb{C}(\mathbb{R})) = 2$ and $\dim(\mathbb{R}(\mathbb{R})) = 1$. Suppose a proper subspace W ($\neq \mathbb{R}$) of $\mathbb{C}(\mathbb{R})$ containing $\mathbb{R}$ exists. Then, we have $1 = \dim(\mathbb{R}(\mathbb{R})) < \dim(W) < \dim(\mathbb{C}(\mathbb{R})) = 2$, a contradiction.

(d) There exists no linear map $T : \mathbb{R}^3 \to \mathbb{R}^2$ such that $T(1,0,3) = (1,1)$ and $T(-2,0,-6) = (2,1)$.
Solution: True. As $T(\alpha x) \neq \alpha T(x)$ for $\alpha = -2$ and $x = (1,0,3)$.

(e) Let $\mathbb{R}^2$ be the standard real inner product space. If $S = \{(-1,-2)\}$, then $S^\perp = \{0\}$.
Solution: false.
$S^\perp = \{(x,y) \in \mathbb{R}^2 : -x - 2y = 0\} = Span\{(-2,1)\}$.

(f) If A is an $n \times n$ matrix with complex entries and $A^k = I$ for some integer $k > 0$, then A has a basis of eigenvectors.

Solution: True.
Consider $f(x) = x^k - 1 = (x - \omega_1)(x - \omega_2) \cdots (x - \omega_k)$.
If $m(x)$ is the minimal polynomial of A, then $m(x)$ divides $f(x)$ and hence it is the product of linear factors.
This implies A is diagonalizable, therefore A has a basis of eigenvectors.

(g) Let V be a real inner product space and $x, y \in V$. If $\|x + y\| = \|x\| + \|y\|$, then x or y is a scalar multiple of the other.
Solution: True.
Taking square of $\|x + y\| = \|x\| + \|y\|$, we get $\langle x, y \rangle = \|x\|\|y\|$.
By Cauchy-Schwartz inequality, the set $\{x, y\}$ is LD.

(h) The matrix $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 4 & -2 \\ -1 & -2 & -4 \end{bmatrix}$ is positive definite.
Solution: False. $|A| = -36 < 0$.

2. Provide a short answer to the following questions. [12]

(a) Let us denote $x = (x_1, x_2, \ldots, x_n) \geq 0$ if $x_i \geq 0$ for each $1 \leq i \leq n$. Suppose $x \geq 0$ whenever $Ax \geq 0$. Show that A is invertible.
Solution:
Consider the homogeneous system $Ax = 0$.
Using hypothesis, we have $x \geq 0$. [1]
Now, we also have $A(-x) = 0$, which by hypothesis gives $-x \geq 0$. Combining both $x \geq 0$ and $-x \geq 0$, we get $x = 0$. So, the homogeneous system $Ax = 0$ has only trivial solution. This implies that A is invertible. [1]

(b) Let M be a 10×10 matrix such that all the entries of M are 5. Find the characteristic polynomial and minimal polynomial of M.

Solution: Since M is symmetric, M is diagonalizable and hence for each eigenvalue λ of M, $AM(\lambda) = GM(\lambda)$. [1/2]
$\because$ rank$(M) = 1$, $\lambda_1 = 0$ is an eigenvalue of M. $GM(0) = 10 - rank(M - 0I) = 9$, so $AM(0) = 9$. [1/2]
If λ_2 is other eigenvalue of M, then $\lambda_2 + 9 \times 0 = trace(M) = 50$.
This gives $\lambda_2 = 50$ with $AM(50) = GM(50) = 1$. [1/2]
So, the characteristic polynomial of M is $x^9(x - 50)$. [1/2]
The minimal polynomial of M is $x(x - 50)$.

(c) Let V be a complex inner product space and A be a linear transformation on A such that $\langle Av, v\rangle = 1$ for all $v \in V$ such that $\|v\| = 1$. Show that $A = I$.

Solution:
Given that for each $v \in V$ with $\|v\| = 1$, $\langle Av, v\rangle = 1 = \langle v, v\rangle$. Equivalently, $\langle (A - I)v, v\rangle = 0$ for all $v \in V$. For any $u, v \in V$
$\langle (A - I)(u + v), u + v\rangle = 0 \implies \langle (A - I)u, v\rangle + \langle (A - I)v, u\rangle = 0.$
$\langle (A - I)(u + iv), u + iv\rangle = 0 \implies -i\langle (A - I)u, v\rangle + i\langle (A - I)v, u\rangle = 0.$
Adding the above two equations, we get $\langle (A - I)u, v\rangle = 0$. [1]
In particular, $\langle (A - I)u, (A - I)u\rangle = 0$. Thus, $(A - I)u = 0$ for all u. Hence, $A = I$. [1]

(d) Let A be a real matrix with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, which are real and positive. How many real matrices B satisfy the equation $A = B^2$.
Solution:

Since A has distinct eigenvalues, it is diagonalizable. Thus, $A = PDP^{-1}$, where $D = diag(\lambda_1, \ldots, \lambda_n)$. [1/2]
Let $B = PD'P^{-1}$, where $D' = diag(\pm\sqrt{\lambda_1}, \ldots, \pm\sqrt{\lambda_n})$. [1/2]
Hence, $A = B^2$, and the number of possible Bs is 2^n. [1]

(e) Find the shortest distance of $(1, 1)$ from the line $y = 4x$ in $\mathbb{R}^2$ under the inner product $\langle (x_1, y_1), (x_2, y_2)\rangle = x_1x_2 + 2x_1y_2 + 2x_2y_1 + 5y_1y_2$.

Solution:
Let W denote the line $y = 4x$.
An orthonormal basis of W is $\{w = (\frac{1}{\sqrt{97}}, \frac{4}{\sqrt{97}})\} = \frac{31}{97}(1, 4)$. [1]
Now, $P_W(v) = \langle (1, 1), w\rangle w = \frac{31}{\sqrt{97}}w$.
Therefore, the shortest distance of $(1, 1)$ from W is
$\|v - P_w(v)\| = ||(1, 1) - \frac{31}{97}(1, 4)|| = ||(\frac{66}{97}, \frac{-27}{97})||$
$= \frac{1}{97}\sqrt{66^2 + 4.66.(-27) + 5.27^2} = \frac{\sqrt{873}}{97}$. [1]

(f) Let A be a 6×6 matrix with characteristic polynomial $p(x) = (x-2)^4(x-3)^2$ and minimal polynomial $m(x) = (x - 2)^2(x - 3)^2$. If the rank of $(A - 2I)$ is 4, write down all possible Jordan canonical forms of A.
Solution: The possible Jordan canonical form J of A is given below:

$$J = \left(\begin{array}{cc|cc|cc} 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{array}\right) \qquad [2]$$

3. For the below matrix A, find an orthogonal matrix P such that $P^T AP$ is diagonal, where P^T denotes the transpose of P. Also, write the matrix A as the linear sum of projection matrices. [6]

$$A = \begin{bmatrix} -1 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Solution:

The characteristic polynomial of A is $-(\lambda - 3)^2(\lambda + 2)$

eigenvalues of A are $\lambda_1 = -2,\ \lambda_2 = \lambda_3 = 3$ [1/2+1/2]

For the eigenvalue $\lambda_1 = -2$, $(A + 2I)X = 0$ gives independent eigenvector $(-2, 1, 0)$. Dividing by its norm we get, $X_1 = \frac{1}{\sqrt{5}}(-2, 1, 0)$. [1]

For the eigenvalue $\lambda_2 = 2$, $(A - 3I)X = 0$ gives independent eigenvectors $(1, 2, 0)$, $(0, 0, 1)$. Dividing by their norms we get, $X_2 = \frac{1}{\sqrt{5}}(1, 2, 0)$ and $X_3 = (0, 0, 1)$. [1+1]

Thus $P = \begin{pmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{5}} \end{pmatrix}$ [1/2]

and $P^T AP = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

This implies that the given matrix is diagonalizable.

Now, $E_1 = \frac{A-\lambda_2 I}{\lambda_1-\lambda_2} = \frac{1}{5}(3I - A)$ and $E_2 = \frac{A-\lambda_1 I}{\lambda_2-\lambda_1} = \frac{1}{5}(A + 2I)$. [1/2+1/2]

then the decomposition of $A = \lambda_1 E_1 + \lambda_2 E_2 = \frac{-2}{5}(3I - A) + \frac{3}{5}(A + 2I)$. [1/2]

4. Find the singular value decomposition of the matrix $M = \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix}$. [4]

Solution:

Given matrix $A = \begin{pmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{pmatrix}$.

Then, $AA^T = \begin{pmatrix} 10 & -20 & -20 \\ -20 & 40 & 40 \\ -20 & 40 & 40 \end{pmatrix}$ and $A^TA = \begin{pmatrix} 81 & -27 \\ -27 & 9 \end{pmatrix}$.

The eigenvalues of A^TA is $\lambda_1 = 90, \ \lambda_2 = 0$. [1/2+1/2]

The eigenvector corresponding to $\lambda_1 = 90$ is $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$.

The eigenvector corresponding to $\lambda_2 = 0$ is $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

Therefore, after orthonormalization of the above vectors, $V = \begin{pmatrix} \frac{-3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{pmatrix}$. [1/2+1/2]

The eigenvalues of AA^T is $\lambda_1 = 90, \ \lambda_2 = 0, \ \lambda_3 = 0$.

The eigenvector corresponding to $\lambda_1 = 90$ is $\begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$.

The eigenvector corresponding to $\lambda_2 = 0$ is $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$.

Applying Gram-Schmidt orthogonalization process, we get $\left\{ \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}, \right.$

$\left. \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{2}{5} \\ \frac{-4}{5} \\ 1 \end{pmatrix} \right\}$.

Therefore, after orthonormalization of the above vectors, $U = \begin{pmatrix} \frac{1}{3} & \frac{2}{\sqrt{5}} & \frac{2}{3\sqrt{5}} \\ \frac{-2}{3} & \frac{1}{\sqrt{5}} & \frac{-4}{3\sqrt{5}} \\ \frac{-2}{3} & 0 & \frac{\sqrt{5}}{3} \end{pmatrix}$.

[1/2+1/2+1/2]

Further, $\Sigma = \begin{pmatrix} 3\sqrt{10} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$. Then $A = U\Sigma V^T$. [1/2]

5. Find a straight line which fits best the given points $(1,2),(2,3),(3,5)$, and $(4,7)$. [4]
Solution:
Given data is $(1,2),(2,3),(3,5),(4,7)$. Let the equation of the line passing through the points be $y = mx + c$. Then,

$$\begin{aligned} 2 &= m + c \\ 3 &= 2m + c \\ 5 &= 3m + c \\ 7 &= 4m + c.[1] \end{aligned}$$

In matrix form, $AX = b$, where $A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 2 \\ 3 \\ 5 \\ 7 \end{pmatrix}$ and $x = \begin{pmatrix} m \\ c \end{pmatrix}$.

$A^T A = \begin{pmatrix} 30 & 10 \\ 10 & 4 \end{pmatrix}$, [1/2]

$A^T b = \begin{pmatrix} 51 \\ 17 \end{pmatrix}$. [1]

Now solving the system $A^T AX = A^T b$, we get $m = \frac{17}{10}$ and $c = 0$. [1+1]

6. For the below matrix A, find a matrix P and J such that $P^{-1}AP = J$, where J denotes the Jordan canonical form of A. [6]

$$A = \begin{bmatrix} 2 & -1 & 0 & 1 \\ 0 & 3 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 3 \end{bmatrix}$$

Solution:

Given matrix $A = \begin{pmatrix} 2 & -1 & 0 & 1 \\ 0 & 3 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 3 \end{pmatrix}$.

The characteristic polynomial is $\det(A - \lambda I) = \det \begin{pmatrix} 2-\lambda & -1 & 0 & 1 \\ 0 & 3-\lambda & -1 & 0 \\ 0 & 1 & 1-\lambda & 0 \\ 0 & -1 & 0 & 3-\lambda \end{pmatrix} =$ $(\lambda - 2)^3(\lambda - 3)$. [1+1]

Now, $N(A - 3I) = N \begin{pmatrix} -1 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

$N(A - 2I) = N \begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$.

Thus, the Jordan Canonical form is $J = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$. [2]

To find a matrix Q such that $Q^{-1}AQ = J$. Then $Q = [X_1 \ X_2 \ X_3 \ X_4]$, where $X_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, $X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $X_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $(A - 2I)X_3 = X_2$. Therefore, $X_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}$.
[1/2+1/2+1/2+1/2]