Indian Institute of Information Technology Allahabad
Convex Optimization (SMAT430C)
Quiz 02: Tentative Marking Scheme

Maximun marks is **25**. If you get more than 25, the extra mark(s) will be added to your total marks out of 150 as a ***bonus***.

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1. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function. Let $c \in \mathbb{R}$ be a point such that for $t \leq c$, f is decreasing, and for $t \geq c$, f is increasing. Prove that f is quasiconvex. (Note: f need not be convex). [4]

   Solution. Let $x, y \in S_\alpha = \{x : f(x) \leq \alpha\}$ such that $x < y$. Then $f(x) \leq \alpha,\ f(y) \leq \alpha$. [1]

   Let $z = \lambda x + (1-\lambda)y$, for $\lambda \in (0,1)$. We have $x < z < y$. We want to show that S_α is convex, i.e., $z \in S_\alpha$ or $f(z) \leq \alpha$. [1]

   If $z \leq c$, then $f(z) \leq f(x) \leq \alpha$ ($\because f$ is decreasing for $t \leq c$). [1]

   If $z \geq c$, then $f(z) \leq f(y) \leq \alpha$ ($\because f$ is increasing for $t \geq c$). [1]

2. Find $\sup\{a^T x : ||x||_2 \leq 5\}$, where a is a nonzero vector in $\mathbb{R}^n$. [5]

   Solution. Let $S = \{a^T x : ||x||_2 \leq 5\}$. Then

   $\langle a, x\rangle = a^T x \leq ||a||_2 ||x||_2 \leq 5||a||_2$, (by Cauchy-Schwartz inequality). [2]

   Let $x_0 = \dfrac{5a}{||a||_2}$. Then $||x_0||_2 = 5$ and, [1]

   $a^T x_0 = \dfrac{a^T(5a)}{||a||_2} = \dfrac{5||a||_2^2}{||a||_2} = 5||a||_2.$ [1]

   $\therefore\ \sup S = 5||a||_2$. [1]

3. Let the pair x and (λ, ν) be primal and dual feasible respectively. If the duality gap associated with this pair is zero, prove that x is primal optimal and (λ, ν) is dual optimal. [7]

   Solution. If x is primal feasible, and (λ, ν) is dual feasible, then

   $f_0(x) - p^* \leq f_0(x) - g(\lambda, \nu)$. [1]

   Given that the duality gap is zero, i.e., $f_0(x) - g(\lambda, \nu) = 0$ or $f_0(x) = g(\lambda, \nu)$. [1]

   $\therefore\ f_0(x) - p^* \leq 0 \implies f_0(x) = p^*$, by definition of p^*. [1]

   $\therefore\ x$ is primal optimal. [1]

   Now, we know that $g(\lambda, \nu) \leq d^* \leq p^*$. [1]

   But $f_0(x) = g(\lambda, \nu) \leq d^* \leq p^* = f_0(x) \implies g(\lambda, \nu) = d^*$. [1]

   $\therefore\ (\lambda, \nu)$ is dual optimal. [1]

4. Find the local extreme values of $f(x, y) = 3y^2 - 2y^3 - 3x^2 + 6xy$. [10]

   Since f is differentiable everywhere, it can assume extreme values only where

   $f_x = 6y - 6x = 0$, and $f_y = 6y - 6y^2 + 6x = 0$. [1+1]

Therefore, the two critical points are $(0,0)$ and $(2,2)$. [1+1]

To classify the critical points, we calculate the second derivatives:

$f_{xx} = -6,\ f_{yy} = 6 - 12y,\ f_{xy} = 6.$ [1+1+1]

The discriminant is given by

$f_{xx}f_{yy} - f_{xy}^2 = 72(y-1).$ [1]

At $(0,0)$, the discriminant is negative, so the function has a saddle point at the origin. [1]

At $(2,2)$ the discriminant is positive, and $f_{xx} < 0$, so $(2,2)$ is a point of local maximum. [1]

5. A vector $g \in \mathbb{R}^n$ is a *subgradient* of $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ at $x \in \text{dom } f$ if for all $y \in \text{dom } f$ we have $f(y) \geq f(x) + g^T(y - x)$. If f is convex and differentiable, then its gradient at x is a subgradient.

   A function f is called *subdifferentiable* at x if there exists at least one subgradient at x. The set of subgradients of f at the point x is called the *subdifferential* of f at x, and is denoted by $\delta f(x)$.

   Consider the absolute value function $f(x) = |x|,\ x \in \mathbb{R}$. Find $\delta|x|$. [10]

   Solution. $|x|$ is convex $\forall\ x \in \mathbb{R}$, and differentiable $\forall\ x \in \mathbb{R} \setminus \{0\}$. Hence, subgradient is unique. [1]

   $\delta|x| = 1$ for $x > 0$, and $\delta|x| = -1$ for $x < 0$. [2+2]

   At $x = 0$, the subdifferential is defined by the inequality $|y| \geq gy$ for all $y \in \mathbb{R}$. [2]

   This is satisfied if and only if $g \in [-1, 1]$. [3]

   Thus,

$$\delta|x| = \begin{cases} +1 & x > 0, \\ [-1, 1] & x = 0, \\ -1 & x < 0. \end{cases}$$

6. Consider the problem

$$\begin{array}{ll} \text{minimize} & -xy \\ \text{subject to} & x + y^2 \leq 2, \\ & x,\ y \geq 0. \end{array}$$

   Find the Lagrangian associated with the above problem. Derive tha KKT conditions. [12]

   Solution. The Lagrangian is given by

$$L(x, y, \lambda_1, \lambda_2, \lambda_3) = -xy + \lambda_1(x + y^2 - 2) + \lambda_2(-x) + \lambda_3(-y). \quad [1]$$

The KKT conditions are given by [11]

$$\begin{aligned}
x + y^2 - 2 &\leq 0, \\
-x &\leq 0, \\
-y &\leq 0, \\
\lambda_1 &\geq 0, \\
\lambda_2 &\geq 0, \\
\lambda_3 &\geq 0, \\
\lambda_1(x + y^2 - 2) &= 0, \\
\lambda_2 x &= 0, \\
\lambda_3 y &= 0, \\
-y + \lambda_1 - \lambda_2 &= 0, \\
-x + 2\lambda_1 y - \lambda_3 &= 0.
\end{aligned}$$