Indian Institute of Information Technology Allahabad
Convex Optimization (SMAT430C)
Quiz I: Tentative Marking Scheme

Duration: **45 Minutes** Date: February 14, 2017
Full Marks: 20 Time: 15:30 – 16:15 IST

Attempt all the Questions. Numbers indicated on the right in [ ] are full marks of that particular problem. All the notations used are standard and same as used in lectures. Please be precise in your answer.

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1. State whether the following statements are true or false. In either case write the precise reason in one or two lines. [2+1+1+1]

   (a) A set is convex if and only if it is midpoint convex.
   Answer. ($\Longrightarrow$) True. For $x,\ y \in C$, we have $\frac{1}{2}x + (1-\frac{1}{2})y = \frac{x+y}{2} \in C$. [1]
   ($\Longleftarrow$) False. Take set of rationals. [1]

   (b) The matrix $\begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}$ is positive semidefinite.
   Answer. False. The matrix is not symmetric. [1]

   (c) A finite nonempty set in $\mathbb{R}^n$ is always open.
   Answer. False. A finite nonempty set does not have any interior point. [1]

   (d) Let K be a proper cone, and $\preceq_K$ a generalized inequality. Then $\preceq_K$ is reflexive.
   Answer. True. Any cone contains 0. [1]

2. Find the distance between two parallel hyperplanes $\{x \in \mathbb{R}^n : a^T x = b_1\}$ and $\{x \in \mathbb{R}^n : a^T x = b_2\}$. [2]

   Answer. A line through the origin and parallel to the vector a, $(x = ta, t \in \mathbb{R})$ intersect the hyperplanes at $x_1 = \dfrac{b_1 a}{a^T a}$ and $x_2 = \dfrac{b_2 a}{a^T a}$, respectively. [1]

   The distance is $||x_1 - x_2||_2 = \left|\left|\dfrac{(b_1 - b_2)a}{a^T a}\right|\right|_2 = \dfrac{|b_1 - b_2|}{||a||_2}$. [1]

3. Let C be an affine set and $x \in C$. Prove that $C - x$ is a subspace. [3]

   Answer. Let $v_1,\ v_2 \in C - x$. Then $v_1 = c_1 - x$ and $v_2 = c_2 - x$ for some $c_1,\ c_2 \in C$. [1]

   Now, $v_1 + v_2 = (c_1 + c_2 - x) - x \in C - x$. ($\because$ C is affine). [1]

   For $v \in C - x$ and $a \in \mathbb{R}$ we have $v = c - x$ for some $c \in C$. Then

   $av = (ac + (1-a)x) - x \in C - x$. [1]

4. Find minimum and minimal element(s) of the set $\{x \in \mathbb{R}^2 : ||x||_2 \leq 1\}$. [3]

   Answer. Let $B = \{x \in \mathbb{R}^2 : ||x||_2 \leq 1\}$.

   The set B doesn't have any minimum element because $x \in B$ is the minimum element of B if all other points of B lie above and to the right of x, which is not true for any element of B. [1]

   x is a minimal element of B if no other point of B lies to the left and below x. [1]

   This implies that all points $x \in B$ such that $||x||_2 = 1$ and $-1 \leq x_i \leq 0, i = 1,\ 2$ are minimal points. [1]

5. Prove that a closed convex set is the intersection of all halfspaces that contain it. (Hint: Use Separating Hyperplane Theorem). [3]

   Answer. Let C be a closed convex set, and $\mathcal{S} = \bigcap\{\mathcal{H} : \mathcal{H} \text{ is a halfspace, } C \subseteq \mathcal{H}\}$.

   Let $x \in C$, and $\mathcal{H}$ a halfspace containing C. Then $x \in \mathcal{H}$ which implies $x \in \mathcal{S}$. Hence, $C \subseteq \mathcal{S}$. [1]

   For the converse, suppose $\exists\ x \in \mathcal{S} \ni x \notin C$. Since C is closed convex, by Separating Hyperplane Theorem there exists a hyperplane that strictly separates x from C, i.e., there is a halfspace $\mathcal{H}$ containing C but not x. Thus, $x \notin \mathcal{S}$, which is a contradiction. Therefore, $\mathcal{S} \subseteq C$. [2]

6. Find the dual cone of $\{Ax : x \succeq 0\}$, where $A \in \mathbb{R}^{n \times n}$. [4]

   Answer. Let $K = \{Ax : x \succeq 0\}$.

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\begin{aligned}
y \in K^* \quad &\iff \quad z^T y \geq 0,\ \forall\ z \in K && [1]\\
&\iff \quad (Ax)^T y \geq 0,\ \forall\ x \succeq 0 && [1]\\
&\iff \quad x^T A^T y \geq 0,\ \forall\ x \succeq 0 && [1]\\
&\iff \quad A^T y \succeq 0. && [1]
\end{aligned}
$$

   Therefore, $K^* = \{y : A^T y \succeq 0\}$.