Indian Institute of Information Technology Allahabad

End Semester Examination, May 2017
Tentative Marking Scheme

Paper Title: Convex Optimization, Paper Code: SMAT430C

Max Marks: 75 Duration: 3 hours

Attempt each question on a new page, and attempt all the parts of **Q.1** and **Q.7** at the same place. Numbers indicated on the right in [ ] are full marks of that particular problem. All notations are standard and same as used in class. A matrix A is indefinite if neither A nor $-A$ is positive semidefinite.

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1. Pick out the correct option(s) in the following questions. No justification is required. [9]

   Note: 1 mark will be awarded for each correct option. However, if any option is wrong you will be awarded 0 mark even if some of the options are correct.

   i) Let $S = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_i \leq 1,\ i = 1, 2\}$ and $D = \{x \in \mathbb{R}^2 : ||x||_2 \leq 1\}$. Then [2]

   a) $S \cap D$ is convex b) $S \cup D$ is convex c) $S \setminus D$ is convex d) None

   Solution. (a) and (b).

   ii) Let $C = \{(1, 0), (1, 1), (-1, -1), (0, 0)\}$. Then [1]

   a) $(0, -1/3) \in \text{conv } C$ b) $(0, 1/3) \in \text{conv } C$

   c) $(0, 1/3)$ is in the conic hull of C d) None

   Solution. (a).

   iii) Consider the set $S = \{(0, 2), (1, 1), (2, 3), (1, 2), (4, 0)\}$. Then [1]

   a) $(0, 2)$ is the minimum element of S b) $(0, 2)$ is a minimal element of S

   c) $(2, 3)$ is a minimal element of S d) None

   Solution. (b).

   iv) Let $K = \{(x_1, x_2) : 0 \leq x_1 \leq x_2\}$. Then [1]

   a) $(1, 3) \preceq_K (3, 4)$ b) $(-1, 2) \in K^*$ c) None

   Solution. (b).

   v) Which of the following statement(s) is correct. [3]

   a) The function $f(x) = \max\{1/2, x, x^2\}$ is convex.

   b) The square of a convex nonnegative function is convex.

   c) The function $f(x) = 1/(1 - x^2)$, with **dom** $f = (-1, 1)$ is log-convex.

   d) None

   Solution. (a), (b) and (c).

   vi) $f(x, y) = \dfrac{x}{y} + \dfrac{y}{x}$ is a posynomial function (x and y are positive variables). [1]

   a) True b) False

   Solution. (a).

2. Classify the following matrix as positive definite, positive semidefinite, indefinite:

$$\begin{pmatrix} 1 & 0 & 4 \\ 0 & 2 & 0 \\ 4 & 0 & 18 \end{pmatrix}.$$

providing justification to your answer. [4]

Solution. Denote the above matrix by A. For $X = (x, y, z) \in \mathbb{R}^3$,

$$X^T AX = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 2 & 0 \\ 4 & 0 & 18 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+4z & 2y & 4x+18z \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad [1]$$

$$= x^2 + 2y^2 + 18z^2 + 8xz. \quad [1]$$

$$= 2y^2 + (x+4z)^2 + 2z^2 > 0, \text{ for } X \neq 0. \quad [1]$$

Therefore, the matrix is positive definite. [1]

Alternative Solution: A is symmetric, i.e., $A^T = A$. [1]

The eigenvalues of A are: $\lambda_1 = 2 > 0$, $\lambda_2 = \frac{19+\sqrt{353}}{2} > 0$, $\lambda_3 = \frac{19-\sqrt{353}}{2} > 0$, [1+1]

$\therefore A$ is positive definite. [1]

Alternative Solution: A is symmetric, i.e., $A^T = A$. [1]

We compute all principal minors of A.

First order principal minors: $|a_{11}| = 1 > 0$, $|a_{22}| = 2 > 0$, $|a_{33}| = 18 > 0$ [3/4]

Second order principal minors: $\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = 2 > 0, \begin{vmatrix} 1 & 4 \\ 4 & 18 \end{vmatrix} = 2 > 0, \begin{vmatrix} 2 & 0 \\ 0 & 18 \end{vmatrix} = 36 > 0.$ [3/4]

Third order principal minors: $\det A = \begin{vmatrix} 1 & 0 & 4 \\ 0 & 2 & 0 \\ 4 & 0 & 18 \end{vmatrix} = 4 > 0.$ [1/2]

$\therefore A$ is positive definite. [1]

3. Consider the halfspace C and hyperbolic set D described below:

$$C = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \leq 0\}$$

and

$$D = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 x_2 \geq 1,\ x_1 \geq 0,\ x_2 \geq 0\}.$$

Prove that C and D can be separated by a hyperplane without finding any explicit expression of a hyperplane separating C and D. State whether they can be strictly separated (no proof required). [8]

Solution. We need to prove that C and D are convex and disjoint, and hence Separating hyperplane theorem can be applied. [1]

Clearly C and D are disjoint. [1]

C is a halfspace and hence convex. [1]

For Convexity of D, let (x_1, x_2), $(y_1, y_2) \in D$, $0 \leq \alpha \leq 1$. Then $\alpha(x_1, x_2) + (1-\alpha)(y_1, y_2) = (\alpha x_1 + (1-\alpha)y_1, \alpha x_2 + (1-\alpha)y_2)$. Now, $[(\alpha x_1 + (1-\alpha)y_1][\alpha x_2 + (1-\alpha)y_2)] = \alpha^2 x_1 x_2 + (1-\alpha)^2 y_1 y_2 + \alpha(1-\alpha)(x_1 y_2 + x_2 y_1) = z$ (say). [1]

Consider two cases:

If $x_1 y_2 + x_2 y_1 \geq x_1 x_2 + y_1 y_2 \geq 2$, then $z \geq \alpha^2 + (1-\alpha)^2 + 2\alpha(1-\alpha) \geq 1$. [1]

If $x_1 y_2 + x_2 y_1 \leq x_1 x_2 + y_1 y_2$, then $(x_1 - y_1)(x_2 - y_2) \geq 0$. This implies that $x_1 \geq y_1,\ x_2 \geq y_2$ or $x_1 \leq y_1,\ x_2 \leq y_2$. The later case is similar to the former one if we interchange the points (x_1, x_2) and (y_1, y_2). [1]

If $x_1 \geq y_1,\ x_2 \geq y_2$, then $x_1 y_2 \geq y_1 y_2 \geq 1$ and $x_2 y_1 \geq y_1 y_2 \geq 1$. Therefore, $z \geq 1$. [1]

They cannot be strictly separated. [1]

4. Show that the set $K = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1| \leq x_2\}$ is a convex cone. Find the dual cone of K. [8]

Solution. Let $(x_1, x_2),\ (y_1, y_2) \in K \implies |x_1| \leq x_2$ and $|y_1| \leq y_2$. [1]

Let $\theta_1, \theta_2 \geq 0$. Then $\theta_1(x_1, x_2) + \theta_2(y_1, y_2) = (\theta_1 x_1 + \theta_2 y_1, \theta_1 x_2 + \theta_2 y_2)$. [1]

Now, $|\theta_1 x_1 + \theta_2 y_1| \leq \theta_1 |x_1| + \theta_2 |y_1| \leq \theta_1 x_2 + \theta_2 y_2$. [1]

Therefore, K is a convex cone.

$K^* = \{y = (y_1, y_2) : x^T y = x_1 y_1 + x_2 y_2 \geq 0,\ \forall\ x = (x_1, x_2) \in K\}$. [1]

We claim that $K^* = K$.

Let $(x_1, x_2) \in K \implies |x_1| \leq x_2$. For any $(y_1, y_2) \in K$, we have $|y_1| \leq y_2$ and [1]

$x_1 y_1 + x_2 y_2 \geq (-x_2)(-y_2) + x_2 y_2 \geq 0$, since $x_2,\ y_2 \geq 0$. [1]

$\implies (x_1, x_2) \in K^*$. Therefore, $K \subseteq K^*$.

Let $(x_1, x_2) \in K^* \implies x_1 y_1 + x_2 y_2 \geq 0,\ \forall\ (y_1, y_2) \in K$.

Choose $(y_1, y_2) = (1, 1) \implies -x_2 \leq x_1$, [1]

if $(y_1, y_2) = (-1, 1) \implies x_1 \leq x_2$. [1]

$\therefore |x_1| \leq x_2 \implies (x_1, x_2) \in K$. Therefore $K^* \subseteq K$.

5. Let

$$u_\alpha(x) = \frac{x^\alpha - 1}{\alpha},\ 0 < \alpha \leq 1,\ u_0(x) = \log x.$$

Here, **dom** $u_\alpha = \mathbb{R}_+$ and **dom** $u_0 = \mathbb{R}_{++}$.

(a) Show that for $x > 0$, $u_0(x) = \lim_{\alpha \to 0} u_\alpha(x)$. [2]

Solution. $\lim_{\alpha \to 0} u_\alpha(x) = \lim_{\alpha \to 0} \dfrac{x^\alpha \log x}{1} = \log x$, (using L'Hopital's Rule). [2]

(b) Show that u_α are concave, monotone increasing, and all satisfy $u_\alpha(1) = 0$. [5]

Solution. It is clear that $u_\alpha(1) = 0$. [1]

$u'_\alpha(x) = x^{\alpha - 1} \geq 0,\ \forall\ x$. [1]

$\implies u_\alpha$ is increasing. [1]

$u''_\alpha(x) = (\alpha - 1)x^{\alpha - 2} \leq 0,\ (\because 0 \leq \alpha < 1)$. [1]

$\implies u_\alpha$ is concave. [1]

6. Consider a network of n nodes, with directed links connecting each pair of nodes. The variables in the problem are the flows on each link: x_{ij} will denote the flow from node i to node j. The cost of the flow along the link from node i to node j is given by $c_{ij} x_{ij}$, where c_{ij} are given constants. The total cost across the network is

$$C = \sum_{i,j=1}^{n} c_{ij} x_{ij}.$$

Each link flow x_{ij} is also subject to a given lower bound l_{ij} and an upper bound u_{ij}.

The external supply at node i is given by b_i, where $b_i > 0$ means an external flow enters the network at node i, and $b_i < 0$ means that at node i, an amount $|b_i|$ flows out of the network. We assume that $\mathbf{1}^T b = 0$, i.e., the total external supply equals total external demand. At each node we have conservation of flow: the total flow into node i along links and the external supply, minus the total flow out along the links, equals zero. The problem is to minimize the total cost of flow through the network, subject to the constraints described above. Formulate this problem as a Linear Program. [5]

Solution. This problem can be formulated as the LP:

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{i,j=1}^{n} c_{ij} x_{ij} \qquad [1] \\ \text{subject to} & b_i + \displaystyle\sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n} x_{ji} = 0,\ i = 1, \ldots, n \qquad [2] \\ & l_{ij} \leq x_{ij} \leq u_{ij}. \qquad [2] \end{array}$$

7. Consider the optimization problem [20]

$$\begin{array}{ll} \text{minimize} & x^2 + 1 \\ \text{subject to} & (x-2)(x-4) \le 0. \end{array}$$

(a) Is the problem convex. Give the feasible set. [3]

Solution. Let $f_0(x) = x^2 + 1$, and $f_1(x) = (x-2)(x-4)$.

Both f_0 and f_1 are convex as $f_0(x)^{''} = f_1(x)^{''} = 2 > 0 \Longrightarrow$ the problem is convex. [2]

The feasible set is $D = [2, 4]$. [1]

(b) Is Slater's condition satisfied. [1]

Solution. Slater's condition is satisfied as $\exists\, x \in (2, 4)$ such that $f_1(x) < 0$. [1]

(c) Find the primal optimal value, and primal optimal point(s). [2]

Solution. The primal optimal point $x^* = 2$, and primal optimal value $p^* = 5$. [2]

(d) Find the Lagrangian and the Lagrange dual function. [7]

Solution. The Lagrangian is $L(x, \lambda) = x^2 + 1 + \lambda(x-2)(x-4) = (1+\lambda)x^2 - 6\lambda x + (1+8\lambda)$. [1]

The Lagrange dual function is $g(\lambda) = \inf_x L(x, \lambda)$. [1]

$L(x, \lambda)' = 2(1+\lambda)x - 6\lambda = 0$ or $x = \frac{3\lambda}{1+\lambda}$. [1]

$L(x, \lambda)'' = 2(1+\lambda) > 0$ if $\lambda > -1$. [1]

Therefore, $L(x, \lambda)$ reaches its minimum at $x = \frac{3\lambda}{1+\lambda}$ provided $\lambda > -1$. [1]

Thus,

$$g(\lambda) = \begin{cases} \dfrac{-9\lambda^2}{(1+\lambda)} + 1 + 8\lambda, & \lambda > -1, \\ -\infty, & \lambda \le -1. \end{cases}$$ [1+1]

(e) State the dual problem. [2]

Solution. The dual problem is

$$\begin{array}{ll} \text{maximize} & \dfrac{-9\lambda^2}{(1+\lambda)} + 1 + 8\lambda \\ \text{subject to} & \lambda \ge 0. \end{array}$$ [1] [1]

(f) Find the dual optimal value, and dual optimal point(s). [3]

Solution. $g(\lambda)' = \dfrac{-18\lambda}{1+\lambda} + \dfrac{9\lambda^2}{(1+\lambda)^2} + 8 = 0$ or $\lambda = 2$. [1]

Thus, dual optimal point $\lambda^* = 2$, and dual optimal value $d^* = 5$. [2]

(g) Verify that strong duality holds. Can you conclude this directly. [2]

Solution. Since, $d^* = p^* \Longrightarrow$ strong duality holds. [1]

We can directly conclude that strong duality holds because Slater's condition is satisfied. [1]

8. Consider the constrained minimization problem

$$\begin{array}{ll} \text{minimize} & (x_1 - 1)^2 + x_2 - 2 \\ \text{subject to} & x_2 - x_1 = 1 \\ & x_1 + x_2 \le 2. \end{array}$$

Write the Karush-Kuhn-Tucker conditions for the above problem. Further, use these conditions to find the optimal point(s) and optimal value. [14]

Solution. The Lagrangian is given by

$$L(x_1, x_2, \lambda, \nu) = (x_1 - 1)^2 + x_2 - 2 + \lambda(x_1 + x_2 - 2) + \nu(x_2 - x_1 - 1).$$

The KKT conditions are [6]

$$\begin{aligned} x_1 + x_2 - 2 &\leq 0, \\ x_2 - x_1 - 1 &= 0, \\ \lambda &\geq 0, \\ \lambda(x_1 + x_2 - 2) &= 0, \\ 2(x_1 - 1) + \lambda - \nu &= 0, \\ 1 + \lambda + \nu &= 0. \end{aligned}$$

Let $\lambda > 0$. Then $x_1 + x_2 - 2 = 0$. Solving above equations we obtain [1]
$x_1 = \frac{1}{2},\ x_2 = \frac{3}{2},\ \nu = -1,\ \lambda = 0$. This contradicts our assumption that $\lambda > 0$. [$2\frac{1}{2}$]
Assume $\lambda = 0$. Solving above equations we get [1]
$x_1 = \frac{1}{2},\ x_2 = \frac{3}{2},\ \nu = -1$. [$1\frac{1}{2}$]
Since the problem is convex and the point $x^* = (x_1^*, x_2^*) = (\frac{1}{2}, \frac{3}{2})$ satisfy the KKT conditions, we conclude that x^* is primal optimal, and optimal value is $p^* = -\frac{1}{4}$. [2]