Indian Institute of Information Technology Allahabad

End Semester Examination, May 2017

Date of Examination (Meeting): 05.05.2017 (1st meeting)

Program Code & Semester: B.Tech. (IT) & Dual Degree B.Tech.-M.Tech. IV Semester

Paper Title: Convex Optimization, Paper Code: SMAT430C

Paper Setter: Abdullah Bin Abu Baker & Anand Kumar Tiwari

Max Marks: 75 Duration: 3 hours

Attempt each question on a new page, and attempt all the parts of **Q.1** and **Q.7** at the same place. Numbers indicated on the right in [ ] are full marks of that particular problem. All notations are standard and same as used in class. A matrix A is indefinite if neither A nor $-A$ is positive semidefinite.

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1. Pick out the correct option(s) in the following questions. No justification is required. [9]

   i) Let $S = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_i \leq 1,\ i = 1, 2\}$ and $D = \{x \in \mathbb{R}^2 : ||x||_2 \leq 1\}$. Then

   a) $S \cap D$ is convex b) $S \cup D$ is convex c) $S \setminus D$ is convex d) None

   ii) Let $C = \{(1, 0), (1, 1), (-1, -1), (0, 0)\}$. Then

   a) $(0, -1/3) \in \text{conv } C$ b) $(0, 1/3) \in \text{conv } C$

   c) $(0, 1/3)$ is in the conic hull of C d) None

   iii) Consider the set $S = \{(0, 2), (1, 1), (2, 3), (1, 2), (4, 0)\}$. Then

   a) $(0, 2)$ is the minimum element of S b) $(0, 2)$ is a minimal element of S

   c) $(2, 3)$ is a minimal element of S d) None

   iv) Let $K = \{(x_1, x_2) : 0 \leq x_1 \leq x_2\}$. Then

   a) $(1, 3) \preceq_K (3, 4)$ b) $(-1, 2) \in K^*$ c) None

   v) Which of the following statement(s) is correct.

   a) The function $f(x) = \max\{1/2, x, x^2\}$ is convex.

   b) The square of a convex nonnegative function is convex.

   c) The function $f(x) = 1/(1 - x^2)$, with $\textbf{dom}\ f = (-1, 1)$ is log-convex.

   d) None

   vi) $f(x, y) = \dfrac{x}{y} + \dfrac{y}{x}$ is a posynomial function (x and y are positive variables).

   a) True b) False

2. Classify the following matrix as positive definite, positive semidefinite, indefinite:

$$\begin{pmatrix} 1 & 0 & 4 \\ 0 & 2 & 0 \\ 4 & 0 & 18 \end{pmatrix}.$$

   providing justification to your answer. [4]

3. Consider the halfspace C and hyperbolic set D described below:

$$C = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \leq 0\}$$

and

$$D = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 x_2 \geq 1,\ x_1 \geq 0,\ x_2 \geq 0\}.$$

Prove that C and D can be separated by a hyperplane without finding any explicit expression of a hyperplane separating C and D. State whether they can be strictly separated (no proof required). [8]

4. Show that the set $K = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1| \leq x_2\}$ is a convex cone. Find the dual cone of K. [8]

5. Let

$$u_\alpha(x) = \frac{x^\alpha - 1}{\alpha},\ 0 < \alpha \leq 1,\ u_0(x) = \log x.$$

Here, $\textbf{dom}\ u_\alpha = \mathbb{R}_+$ and $\textbf{dom}\ u_0 = \mathbb{R}_{++}$.

(a) Show that for $x > 0$, $u_0(x) = \lim_{\alpha \to 0} u_\alpha(x)$. [2]

(b) Show that u_α are concave, monotone increasing, and all satisfy $u_\alpha(1) = 0$. [5]

6. Consider a network of n nodes, with directed links connecting each pair of nodes. The variables in the problem are the flows on each link: x_{ij} will denote the flow from node i to node j. The cost of the flow along the link from node i to node j is given by $c_{ij}x_{ij}$, where c_{ij} are given constants. The total cost across the network is

$$C = \sum_{i,j=1}^{n} c_{ij} x_{ij}.$$

Each link flow x_{ij} is also subject to a given lower bound l_{ij} and an upper bound u_{ij}.

The external supply at node i is given by b_i, where $b_i > 0$ means an external flow enters the network at node i, and $b_i < 0$ means that at node i, an amount $|b_i|$ flows out of the network. We assume that $\mathbf{1}^T b = 0$, i.e., the total external supply equals total external demand. At each node we have conservation of flow: the total flow into node i along links and the external supply, minus the total flow out along the links, equals zero. The problem is to minimize the total cost of flow through the network, subject to the constraints described above. Formulate this problem as a Linear Program. [5]

7. Consider the optimization problem [20]

$$\begin{array}{ll} \text{minimize} & x^2 + 1 \\ \text{subject to} & (x-2)(x-4) \leq 0. \end{array}$$

(a) Is the problem convex. Give the feasible set.

(b) Is Slater's condition satisfied.

(c) Find the primal optimal value, and primal optimal point(s).

(d) Find the Lagrangian and the Lagrange dual function.

(e) State the dual problem.

(f) Find the dual optimal value, and dual optimal point(s).

(g) Verify that strong duality holds. Can you conclude this directly.

8. Consider the constrained minimization problem

$$\begin{array}{ll} \text{minimize} & (x_1 - 1)^2 + x_2 - 2 \\ \text{subject to} & x_2 - x_1 = 1 \\ & x_1 + x_2 \leq 2. \end{array}$$

Write the Karush-Kuhn-Tucker conditions for the above problem. Further, use these conditions to find the optimal point(s) and optimal value. [14]