Indian Institute of Information Technology Allahabad
Mathematics - I (SMAT130C)
Quiz 02: Tentative Marking Scheme

Duration: 1 Hour
Full Marks: 20

Date: November 26, 2016
Time: 15:00 – 16:00 IST

Attempt all the Questions. Numbers indicated on the right in [ ] are full marks of that particular problem. Please be precise in your answer. You are not allowed to write anything on the question paper.

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1. Determine the values of p for which $\int_0^\infty \frac{1-e^{-x}}{x^p}dx$ converges. [6]

   Solution. Let $I_1 = \int_0^1 \frac{1-e^{-x}}{x^p}dx$ and $I_2 = \int_1^\infty \frac{1-e^{-x}}{x^p}dx$. [1]

   We apply Limit Comparison Test for I_1 and I_2.

   Since $\lim_{x\to 0} \frac{\frac{1-e^{-x}}{x^p}}{\frac{1}{x^{p-1}}} = 1$, by LCT with $\frac{1}{x^{p-1}}$, we see that I_1 is convergent if and only if $p-1 < 1$, that is, $p < 2$. [2]

   For I_2 applying LCT with $\frac{1}{x^p}$ we see that $\lim_{x\to\infty} \frac{\frac{1-e^{-x}}{x^p}}{\frac{1}{x^p}} = 1$. Thus, I_2 is convergent if and only if $p > 1$. [2]

   Therefore, $\int_0^\infty \frac{1-e^{-x}}{x^p}dx$ is convergent if and only if $1 < p < 2$. [1]

2. Sketch the graph of $r^2 = 8\cos 2\theta$. Find the area of the region that lies inside the curve $r^2 = 8\cos 2\theta$ and outside the circle $r = 2$. [Express the answer in an integral expression. There is no need to evaluate the integral]. [4]

   Solution. The graph of the polar equation $r^2 = 8\cos 2\theta$ is given in figure 2(a). [2]

   Solving the equation $8\cos 2\theta = 4$, we get $\theta = \pi/6$. Thus, both curves intersect at $(2, \pi/6)$ in the first quadrant. [1]

   Because of symmetry the area of the region (see figure 2(b)) is given by

   $$4\int_0^{\pi/6} \frac{1}{2}(8\cos 2\theta - 4). \qquad [1]$$

3. Let $f(x, y) = 3x^4 - 4x^2y + y^2$. Show that f has a local minimum at $(0, 0)$ along every line through $(0, 0)$. Is $(0, 0)$ a saddle point for f. [4]

   Solution. Along x-axis, the local minimum of the function is at $(0, 0)$. [1]

   Let $y = mx,\ m \neq 0$. Then $f(x, mx) = 3x^4 - 4mx^3 + m^2x^2$ is a function of one variable. [1]

   As $f''(0, 0) = 2m^2 > 0$, by second derivative test, $(0, 0)$ is local minimum. [1]

   Since, $f(x, y) = (3x^2 - y)(x^2 - y)$, we see that in the region between the parabolas $y = 3x^2$ and $y = x^2$, the function takes negative values and is positive everywhere else. Thus, $(0, 0)$ is a saddle point for f. [1]

4. Let

$$f(x,y) = \begin{cases} \frac{y}{|y|}\sqrt{x^2+y^2}, & \text{if } y \neq 0 \\ 0, & \text{if } y = 0. \end{cases}$$

Show that all directional derivatives of f exist at $(0,0)$, but f is not differentiable at $(0,0)$. [6]

Solution. Let $U = (u_1, u_2) \in \mathbb{R}^2$ such that $||U|| = 1$.

Now,

$$D_{(0,0)}f(U) = \lim_{t\to 0} \frac{f(0+tU) - f(0)}{t} = \lim_{t\to 0} \frac{f(tu_1, tu_2)}{t} = 0 \text{ if } u_2 = 0. \quad [1]$$

If $u_2 \neq 0$, then

$$\begin{aligned} D_{(0,0)}f(U) &= \lim_{t\to 0} \frac{f(tu_1, tu_2)}{t} \\ &= \lim_{t\to 0} \frac{\frac{tu_2}{|tu_2|}\sqrt{t^2u_1^2 + t^2u_2^2}}{t} \\ &= \lim_{t\to 0} \frac{\frac{tu_2}{|tu_2|}\ |t|\ ||U||}{t} & [1] \\ &= \frac{u_2}{|u_2|}. & [1] \end{aligned}$$

Therefore, directional derivatives in all directions exist.

We see that $f_x(0,0) = 0$ and $f_y(0,0) = 1$. If f is differentiable at $(0,0)$ then $f'(0,0) = \alpha = (0,1)$. [1]

Note that

$$\epsilon(h,k) = \frac{\frac{k}{|k|}\sqrt{h^2+k^2} - k}{\sqrt{h^2+k^2}}. \quad [1]$$

If we take $h = k$, then $\epsilon(h,h) = (1 - \frac{1}{\sqrt{2}})\frac{k}{|k|} \nrightarrow 0$ as $k \to 0$. Therefore, $\epsilon(h,k) \nrightarrow 0$ as $(h,k) \to (0,0)$ [1]