Indian Institute of Information Technology Allahabad
Mathematics - I (SMAT130C)
Quiz 01: Tentative Marking Scheme

Duration: 30 Minutes
Full Marks: 20

Date: September 03, 2016
Time: 11:00 – 11:30 IST

Attempt all the Questions. Numbers indicated on the right in [ ] are full marks of that particular problem. All the notations used are standard and same as used in lectures. Please be precise in your answer.

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1. Prove the following statements.

   (a) Let A be a nonempty subset of $\mathbb{R}$, and $\alpha \in \mathbb{R}$ be the least upper bound of A. Then there exists a sequence (a_n) in A such that $a_n \longrightarrow \alpha$. [5]
   Solution. For every $n \in \mathbb{N}$, there exists $a_n \in A$ such that $\alpha - \frac{1}{n} < a_n$, (otherwise $\alpha - \frac{1}{n}$ would be the supremum). [2]
   As α is an upper bound for A, we have $\alpha - \frac{1}{n} < a_n \leq \alpha < \alpha + \frac{1}{n}$. [2]
   Therefore, $a_n \longrightarrow \alpha$. [1]

   (b) Let (x_n) be a sequence of real numbers. If $x_n \longrightarrow x$, then $|x_n| \longrightarrow |x|$. Is the converse true? [3]
   Solution. Let $\epsilon > 0$. As $x_n \longrightarrow x$, there exists $N \in \mathbb{N}$ such that $|x_n - x| < \epsilon$ for all $n \geq N$. [1]
   Now, $||x_n| - |x|| \leq |x_n - x| < \epsilon$ for all $n \geq N$. [1]
   Therefore, $|x_n| \longrightarrow |x|$.
   The converse is not true. Consider the sequence $x_n = (-1)^n$. [1]

   (c) Let (a_n) be a sequence in $\mathbb{R}$. If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \longrightarrow 0$. [2]
   Solution. Since $\sum_{n=1}^{\infty} a_n$ is convergent, the sequence (S_n) of partial sums is convergent. [1]
   Therefore, $a_n = S_n - S_{n-1} \longrightarrow 0$. [1]

2. Let (x_n) be sequence defined by

$$x_n = n^{\alpha}(1+\beta)^{-n} \sin n$$

   for all $n \in \mathbb{N}$, where α and β are fixed positive real numbers. Show that (x_n) converges. (Don't try with L'Hopital's Rule!). [5]

   Solution. Let $y_n = \frac{n^{\alpha}}{(1+\beta)^n}$. Then $\frac{y_{n+1}}{y_n} = (\frac{n+1}{n})^{\alpha} \frac{1}{1+\beta} \longrightarrow \frac{1}{1+\beta} < 1$. [2]

   Therefore, $y_n \longrightarrow 0$. [1]

   Since, $|x_n| \leq |y_n|$ (or $(\sin n)$ is a bounded sequence), [1]

   we have $x_n \longrightarrow 0$. [1]

3. Let $y \in (0, 1)$. Discuss the convergence/divergence of the series

$$\sum_{n=1}^{\infty} [(n+1)y^n + \sin n]. \qquad [5]$$

Solution. Note that $a_n = (n+1)y^n \longrightarrow 0$ because $\frac{a_{n+1}}{a_n} \longrightarrow y < 1$. [2]

Hence $((n+1)y^n + \sin n) \not\longrightarrow 0$. [2]

Therefore, the series diverges. [1]