Indian Institute of Information Technology Allahabad

Mid Semester Examination, September 2016
Tentative Marking Scheme

Date of Examination (Meeting): 04.10.2016 (2nd meeting)

Program Code & Semester: B.Tech. (IT), B.Tech. (ECE), Dual Degree - Semester I

Paper Title: Mathematics - I, Paper Code: SMAT130C

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Max Marks: 40 Duration: 2 hours

Attempt each question on a new page, and attempt all the parts of a question at the same place. Numbers indicated on the right in [ ] are full marks of that particular problem. All the notations used are standard and same as used in lectures.

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1. (a) Evaluate $\lim_{x \to 0} \frac{xe^x - \log(1+x)}{x^2}$. [2]

Solution. By L'Hôpital's rule,

$$\begin{aligned}\lim_{x\to 0} \frac{xe^x - \log(1+x)}{x^2} &= \lim_{x\to 0} \frac{xe^x + e^x - \frac{1}{1+x}}{2x} \\ &= \lim_{x\to 0} \frac{xe^x + 2e^x + \frac{1}{(1+x)^2}}{2} \\ &= \frac{3}{2}.\end{aligned}$$ [2]

(b) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function such that $f(c) > 0$ for some $c \in \mathbb{R}$. Show that there exists a $\delta > 0$ such that $f(x) > 0$ for all $x \in (c - \delta, c + \delta)$. [2]

Solution. Let $\epsilon = \frac{f(c)}{2}$. By continuity,
$\exists\ \delta > 0 \ni |f(x) - f(c)| < \frac{f(c)}{2}$ whenever $|x - c| < \delta$. [1]

This implies that $f(x) > \frac{f(c)}{2} > 0$ for all $x \in (c - \delta, c + \delta)$. [1]

2. Find the number of real solutions of the equation $x^{17} - e^{-x} + 5x + \cos x = 0$. [4]

Solution. Let $f(x) = x^{17} - e^{-x} + 5x + \cos x = 0$.

Observe that $f(2) > 0$ and $f(-2) < 0$. [1]

By Intermediate value property, $f(x) = 0$ has one real solution. [1]

If there are two distinct roots, then by Rolle's theorem there is some c such that $f'(c) = 0$ which is not possible since $f'(x) > 0, \forall\ x \in \mathbb{R}$. [2]

3. Let f be differentiable on $[a, b]$. Show that there exist $c_1, c_2, c_3 \in (a, b)$ such that $c_2 \neq c_3$ and $f'(c_2) + f'(c_3) = 2f'(c_1)$. [5]

Solution. By MVT, $\exists\ c_1 \in (a, b)$ such that $f(b) - f(a) = f'(c_1)(b - a)$. [1]

By MVT, $\exists\ c_2 \in (a, \frac{a+b}{2})$ such that $f(\frac{a+b}{2}) - f(a) = f'(c_2)(\frac{b-a}{2})$. [1]

and $\exists\ c_3 \in (\frac{a+b}{2}, b)$ such that $f(b) - f(\frac{a+b}{2}) = f'(c_3)(\frac{b-a}{2})$. [1]

This implies that $f(b) - f(a) = \big[f'(c_2) + f'(c_3)\big](\frac{b-a}{2})$ [1]

That is, $f'(c_2) + f'(c_3) = 2f'(c_1)$. [1]

4. Find the intervals of decrease/increase, intervals of concavity/convexity, points of local minima/local maxima, points of inflection for the function $f(x) = \dfrac{2x^2+1}{x^2+1}$. [5]

Solution. $f'(x) = \frac{2x}{(x^2+1)^2} \implies f$ is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$ [1]

and has a local minimum at $x = 0$. [1]

$f''(x) = \frac{2(1-3x^2)}{(x^2+1)^3} \implies f$ is concave on $(-\infty, -\frac{1}{\sqrt{3}})$ and $(\frac{1}{\sqrt{3}}, \infty)$ [1]

and convex on $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ [1]

and $-\frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}$ are the points of inflection. [1]

5. (a) Let $f : [0,1] \longrightarrow \mathbb{R}$ and n be a fixed non-negative integer. Suppose $f^{(n+1)}$ exists on $[0,1]$ and $f^{(n+1)}(x) = 0$ for all $x \in [0,1]$. Show that f is polynomial of degree less than or equal to n. [3]

Solution. Let $x > 0$. By Taylor's theorem, there exists $c \in (0, x)$ such that

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \cdots + \frac{f^{(n)}(0)x^n}{n!} + \frac{f^{(n+1)}(c)x^{n+1}}{(n+1)!}. \quad [2]$$

Since $f^{(n+1)}(x) = 0$ for all $x \in [0,1]$, $f(x)$ is polynomial of degree less than or equal to n. [1]

(b) Show that for $0 \leq x \leq 1$,

$$\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \ \frac{x^n}{n}. \quad [5]$$

Solution. By Taylor's theorem, $\exists\ c \in (0, x)$ such that

$$\log(1+x) = x - \frac{1}{2}x^2 + \cdots + \frac{(-1)^{n+1}}{n}\ x^n + \frac{(-1)^{n+2}}{(n+1)} \frac{x^{n+1}}{(1+c)^{n+1}} \quad [2]$$

Note that $|E_n(x)| = |\frac{(-1)^{n+2}}{(n+1)} \frac{x^{n+1}}{(1+c)^{n+1}}| \leq |\frac{x^{n+1}}{n+1}|$ [2]

Let $a_n = \frac{x^n}{n}$. When $x = 1$, $a_n \longrightarrow 0$. If $x \in (0,1)$, $\frac{a_{n+1}}{a_n} = \frac{x}{1+\frac{1}{n}} \longrightarrow x$. Hence $a_n \longrightarrow 0$. [1]

Therefore, $\sum_{n=1}^{\infty} (-1)^{n+1}\ \frac{x^n}{n} = \log(1+x)$.

6. (a) Let $f : [a,b] \longrightarrow \mathbb{R}$ be a continuous function such that $f(x) \geq 0$ for all $x \in [a,b]$ and $\int_a^b f(x)dx = 0$. Show that $f(x) = 0$ for all $x \in [a,b]$. [4]

Solution. Let $c \in [a,b]$ such that $f(c) > \alpha$ for some $\alpha > 0$. By continuity of f, $\exists$ a $\delta > 0$ such that $(c-\delta, c+\delta) \subseteq (a,b)$ and $f(x) > \alpha$ on $(c-\delta, c+\delta)$. [1]

Let $P = \{a, c-\delta, c+\delta, b\}$ be a partition of $[a,b]$. [1]

Then $\int_a^b f(x)dx \geq L(P,f) > \alpha\delta > 0$, which is a contradiction. [1]

Similarly we can show that $f(a) = 0$ and $f(b) = 0$. [1]

(b) Does there exists an integrable function f on $[a,b]$ such that $f(x) \geq 0$ for all $x \in [a,b]$ and $\int_a^b f(x)dx = 0$ but $f(c) \neq 0$ for some $c \in [a,b]$. [2]

Solution. Yes. Let $f(x) = 0$ for all $x \in (a,b]$ and $f(a) = 1$. Then $\int_a^b f(x)dx = 0$ but $f(a) \neq 0$.

(c) Let $f : [0,1] \longrightarrow \mathbb{R}$ be defined as

$$f(x) = \begin{cases} x, & x \text{ is rational} \\ 0, & x \text{ is irrational.} \end{cases}$$

Evaluate the upper and lower integrals of f and show that f is not integrable. [8]

Solution. Let $P = \{x_0, x_1, \ldots, x_n\}$ be any partition of $[0,1]$. Since there exists an irrational number in each sub-interval $[x_{i-1}, x_i]$, $L(P,f) = 0$, and hence $\displaystyle\int_{\underline{a}}^{b} f(x)dx = 0$. [1]

Now,

$$
\begin{aligned}
U(P,f) &= \sum_{i=1}^{n} x_i(x_i - x_{i-1}) && [1]\\
&= \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} x_{i-1}x_i\\
&\geq \sum_{i=1}^{n} x_i^2 - \frac{1}{2}\sum_{i=1}^{n}(x_{i-1}^2 + x_i^2) \text{ (Using AM-GM inequality)}\\
&= \frac{1}{2}\sum_{i=1}^{n}(x_{i-1}^2 - x_i^2) = \frac{1}{2}. && [1]\\
\implies & \overline{\int_a^b} f(x)dx \geq \frac{1}{2}. && [1]
\end{aligned}
$$

For each $n \in \mathbb{N}$, consider $P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n}, \frac{n}{n} = 1\right\}$. [1]

Then

$$
\begin{aligned}
U(P_n,f) &= \frac{n+1}{2n} = \frac{1}{2} + \frac{1}{2n} && [1]\\
\implies & \inf\left\{U(P_n,f) : n \in \mathbb{N}\right\} = \frac{1}{2} && [1]\\
\implies & \overline{\int_a^b} f(x)dx \leq \inf\left\{U(P_n,f) : n \in \mathbb{N}\right\} = \frac{1}{2}. && [1]
\end{aligned}
$$

Therefore, $\overline{\int_a^b} f(x)dx = \frac{1}{2}$.