Indian Institute of Information Technology Allahabad

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Max Marks: 75 Duration: 3 hours

Attempt each question on a new page, and attempt all the parts of a question at the same place. Numbers indicated on the right in [ ] are full marks of that particular problem. All the notations used are standard and same as used in lectures. Do not write on question paper and cover pages except your detail. This question paper has two pages. Use of Calculator is not allowed.

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1. Provide a short proof or answer of the following statements.

(a) Let $A, B \in \mathbb{R}$ such that $A \subseteq B$. Then $\inf B \leq \inf A$. [2]

Solution.

Let $a = \inf A,\ b = \inf B$.

$\Longrightarrow b \leq x$ for all $x \in B$.

$\Longrightarrow b \leq x$ for all $x \in A$. [1]

$\Longrightarrow b \leq a$. [1]

(b) Show that $\lim_{x\to 0} \cos \frac{1}{x}$ does not exist. [2]

Solution.

Let $x_n = \frac{1}{n\pi}$. Then $x_n \longrightarrow 0$. [1]

But $f(x_n) = \cos n\pi = (-1)^n$, which is not convergent. [1]

Hence, $\lim_{x\to 0} \cos \frac{1}{x}$ does not exist.

(c) Find the radius of convergence of series [2]

$$\frac{1}{2}x + \frac{1.3}{2.5}x^2 + \frac{1.3.5}{2.5.8}x^3 + \cdots.$$

Solution.

The radius of convergence R of the series is

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{1.3.\ldots(2n-1)}{2.5\ldots(3n-1)} \frac{2.5\ldots(3n+2)}{1.3\ldots(2n+1)} = \lim_{n\to\infty} \frac{3n+2}{2n+1} = \frac{3}{2}.$$

[2]

(d) Find the Macluarin series of the function defined by [3]

$$f(x) = \begin{cases} e^{-1/x^2}, & \text{for } x \neq 0 \\ 0, & \text{for } x = 0. \end{cases}$$

Solution.

Using L'Hopital rule,

$$f'(0) = \lim_{x\to 0} \frac{e^{-1/x^2}}{x} = \lim_{x\to\infty} \frac{x}{e^{x^2}} = 0. \qquad [1]$$

Similarly, $f^{(k)}(0) = 0$ for all $k = 2, 3 \ldots$. [1]

Therefore the Macluarin series of f (for any $x \in \mathbb{R}$) is identically zero. [1]

(e) Examine the convergence of the integral $\int_1^\infty e^{-x^2}\ dx$. [2]

Solution.

Since $x^2 < e^{x^2}$ for every $x \in \mathbb{R}$, $e^{-x^2} < \frac{1}{x^2}$. [1]

By comparison test, $\int_1^\infty e^{-x^2}\ dx$ is convergent. [1]

2. Find the limit of the sequence $(\frac{\sin n}{n}, e^{(\sqrt{n}-\sqrt{n+1})}, \log(1+\frac{1}{n^3}))$ in $\mathbb{R}^3$. [5]

Solution.

The sequence $\frac{\sin n}{n} \longrightarrow 0$ and $\log(1+\frac{1}{n^3}) \longrightarrow \log 1 = 0$. [1+1]

As $\sqrt{n}-\sqrt{n+1} = \frac{-1}{\sqrt{n}+\sqrt{n+1}} \longrightarrow 0 \Longrightarrow e^{(\sqrt{n}-\sqrt{n+1})} \longrightarrow e^0 = 1$. [1+1]

Therefore, $(\frac{\sin n}{n}, e^{(\sqrt{n}-\sqrt{n+1})}, \log(1+\frac{1}{n^3})) \longrightarrow (0,1,0)$. [1]

3. (a) Let (x_n) be a bounded sequence. Assume that $x_{n+1} \geq x_n - 2^{-n}$. Show that (x_n) is convergent. [4]

Solution.

Let $y_n = x_n - \frac{1}{2^{n-1}}$. Then (y_n) is bounded. [1]

Now, $y_{n+1} = x_{n+1} - \frac{1}{2^n} \geq x_n - \frac{1}{2^n} - \frac{1}{2^n} = y_n$. [1]

Hence, (y_n) is an increasing sequence and hence convergent. [1]

Therefore, (x_n) is also convergent. [1]

(b) Let $f : [0,1] \longrightarrow \mathbb{R}$ and $a_n := f(\frac{1}{n}) - f(\frac{1}{n+1})$. Prove the following.

i. If f is continuous, then $\sum_{n=1}^{\infty} a_n$ converges. [2]

Solution.

The sequence of partial sums S_n of $\sum_{n=1}^{\infty} a_n$ is $f(1) - f(\frac{1}{n+1})$. [1]

Since, $S_n \longrightarrow f(1) - f(0)$, $\sum_{n=1}^{\infty} a_n$ converges. [1]

ii. If f is differentiable and $|f'(x)| < \frac{1}{2}$, $\forall\ x \in [0,1]$, then $\sum_{n=1}^{\infty} a_n\sqrt{n}\cos n$ converges. [4]

Solution.

By Mean Value Theorem, $a_n = f(\frac{1}{n}) - f(\frac{1}{n+1}) = f'(c)(\frac{1}{n} - \frac{1}{n+1})$, for some $c \in (\frac{1}{n+1}, \frac{1}{n})$. [1]

This implies that

$|a_n\sqrt{n}\cos n| = |f'(c)||\frac{1}{n} - \frac{1}{n+1}||\sqrt{n}\cos n| < \frac{1}{2}\frac{1}{n(n+1)}\sqrt{n} < \frac{1}{n\sqrt{n}}$. [2]

By comparison test, $\sum_{n=1}^{\infty} |a_n\sqrt{n}\cos n|$ converges, and hence $\sum_{n=1}^{\infty} a_n\sqrt{n}\cos n$ converges. [1]

4. The region bounded by the functions $y = x^2 + x + 1$, $y = 1$ and $x = 1$ is revolved about the line $x = 2$. Find the volume of the solid generated by the shell method. [4]

Solution.

For each x from 0 to 1, we consider a shell.

The shell radius at x is $2 - x$ and the shell height is $x^2 + x$, see Figure 4. [2]

Therefore, the volume of the solid generated is $\int_0^1 2\pi(2-x)(x^2+x)\ dx = 13\pi/6$. [2]

5. Find the length of the curve [3]

$$y = \frac{4\sqrt{2}}{3}x^{3/2} - 1, \quad 0 \leq x \leq 1.$$

Solution.

$$\frac{dy}{dx} = 2\sqrt{2}x^{1/2}.$$

[1]

The length of the curve from $x = 0$ to $x = 1$ is $\int_0^1 \sqrt{1 + (\frac{dy}{dx})^2}\ dx = \int_0^1 \sqrt{1+8x}\ dx$ [1]

$= 13/6$. [1]

6. The curve $x(t) = 2\cos t - \cos 2t$, $y(t) = 2\sin t - \sin 2t$, $0 \leq t \leq \pi$ is revolved about the x-axis. Calculate the area of the surface generated. [4]

Solution.

Observe that $x'(t)^2 + y'(t)^2 = 8(1-\cos t)$. [1]

The surface area is

$$\begin{aligned}
\int_0^\pi 2\pi y(t)\sqrt{x'(t)^2+y'(t)^2}\,dt &= 2\pi\int_0^\pi 2\sin t\,(1-\cos t)\,2\sqrt{2}\sqrt{1-\cos t}\,dt && [1]\\
&= 8\pi\sqrt{2}\int_0^\pi \sin t\,(1-\cos t)^{3/2}\,dt && [1]\\
&= \frac{128\pi}{5}. && [1]
\end{aligned}$$

7. Consider the function

$$f(x,y) = \begin{cases} \frac{3x^2y-y^3}{x^2+y^2}, & \text{if } (x,y) \neq (0,0)\\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

Answer the following.

(a) Discuss the continuity of f at $(0,0)$. [2]

Solution.

$|f(x,y) - f(0,0)| = |\frac{y(3x^2-y^2)}{x^2+y^2}| \leq |\frac{y(3x^2+3y^2)}{x^2+y^2}| = |3y| \longrightarrow 0$ as $(x,y) \longrightarrow 0$.

Thus f is continuous at $(0,0)$. [2]

(b) Evaluate $f_y(x,0)$ for $x \neq 0$. [2]

Solution.

$$\begin{aligned}
f_y(x,0) &= \lim_{h\to 0}\frac{f(x,h)-f(x,0)}{h} && [1]\\
&= \lim_{h\to 0}\frac{3x^2-h^2}{x^2+h^2} = 3. && [1]
\end{aligned}$$

(c) Is f_y continuous at $(0,0)$. [3]

Solution.

$f_y(0,0) = \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h} = -1.$ [1]

Since $f_y(x,0) \nrightarrow f_y(0,0)$ as $x \to 0$, f_y is not continuous at $(0,0)$. [2]

(d) Find the directional derivative of f at $(0,0)$ in the direction of $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. [2]

Solution. The directional derivative of f at $(0,0)$ in the direction of $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ is

$$\begin{aligned}
D_{(0,0)}f(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) &= \lim_{t\to 0}\frac{f((0,0)+t(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})) - f(0,0)}{t} && [1]\\
&= \lim_{t\to 0}\frac{\frac{t^3}{\sqrt{2}}}{t^3} = \frac{1}{\sqrt{2}} && [1]
\end{aligned}$$

(e) Discuss the differentiability of f at $(0,0)$. [3]

Solution. $f_x(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = 0.$ [1]

Let $H = (h,k) \in \mathbb{R}^2$. Then the error function is

$$\begin{aligned}
\epsilon(H) &= \frac{f(h,k) - f(0,0) - hf_x(0,0) - kf_y(0,0)}{||H||}\\
&= \frac{4h^2k}{(h^2+k^2)^{\frac{3}{2}}}. && [1]
\end{aligned}$$

Take $h = k$. Then $\epsilon(h,h) = \sqrt{2} \nrightarrow 0$ as $h \to 0$. Therefore, f is not differentiable at $(0,0)$. [1]

Alternatively

$f_x(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = 0.$ [1]

If f is differentiable at (a,b), then $D_{(a,b)}f(U) = (f_x(a,b), f_y(a,b)) \cdot U$. [1]

Since $D_{(0,0)}f(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = \frac{1}{\sqrt{2}}$, but

$(f_x(0,0), f_y(0,0)) \cdot (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = (0,-1)\cdot(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = -\frac{1}{\sqrt{2}}$, f is not differentiable at $(0,0)$. [1]

8. Evaluate the following integrals:

(a) $\int_0^1 \int_{x^2}^1 x^3 e^{y^3} \, dy \, dx.$ [4]

Solution. See Figure 8(a).

$$\begin{aligned} \int_0^1 \int_{x^2}^1 x^3 e^{y^3} dy \, dx &= \int_0^1 \int_0^{\sqrt{y}} x^3 e^{y^3} dx \, dy \quad [2] \\ &= \int_0^1 \frac{1}{4} y^2 e^{y^3} \, dy \quad [1] \\ &= \frac{1}{12}(e-1). \quad [1] \end{aligned}$$

(b) $\iiint_D \frac{z}{(x^2+y^2+z^2)^{3/2}} \, dx \, dy \, dz$; where D is the region bounded above by the sphere

$x^2+y^2+z^2 = 2$ and below by the plane $z = 1$. [7]

Solution.

The given sphere is of radius $\sqrt{2}$.

If we allow ϕ to vary independently, then ϕ varies from 0 to $\pi/4$, see Figure 8(b'). [1]

If we fix ϕ and allow θ to vary from 0 to 2π then we obatain a surface of a cone, see Figure 8(b). [1]

Since only a part of the cone is lying in the given region, for a fixed ϕ and θ, ρ varies from $\sec\phi$ to $\sqrt{2}$, see Figure 8(b). [1]

Therefore,

$$\begin{aligned} \iiint_D \frac{z}{(x^2+y^2+z^2)^{3/2}} dx \, dy \, dz &= \int_0^{\pi/4} \int_0^{2\pi} \int_{\sec\phi}^{\sqrt{2}} \frac{\cos\phi}{\rho^2} |J(\rho,\phi,\theta)| \, d\rho \, d\theta \, d\phi \\ &= \int_0^{\pi/4} \int_0^{2\pi} \int_{\sec\phi}^{\sqrt{2}} \frac{\cos\phi}{\rho^2} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi \quad [2] \\ &= 2\pi \int_0^{\pi/4} (\sqrt{2}\sin\phi\cos\phi - \sin\phi) d\phi \quad [1] \\ &= \left(\frac{3}{\sqrt{2}} - 2 \right) \pi. \quad [1] \end{aligned}$$

9. Let $f(x,y) = (xy^2, x^2y + 2x)$ and C be any square in the plane. Show that the line integral of f along C depends on the area of the square and not on its location in the plane. [3]

Solution. Let R be a square enclosed by the boundary C. Then by Green's Theorem

$$\begin{aligned} \int_C xy^2 dx + (x^2y+2x)dy &= \iint_R \left(\frac{\partial}{\partial x}(x^2y+2x) - \frac{\partial}{\partial y} xy^2 \right) \, dxdy \quad [1] \\ &= \iint_R (2xy + 2 - 2xy) dx \, dy \\ &= 2 \iint_R dx \, dy \quad [1] \\ &= 2Area(R). \quad [1] \end{aligned}$$

Thus the value of $\int_C xy^2 dx + (x^2y+2x)dy$ around any square depends only on the size of the square C and not on its location in the plane.

10. Find the absolute maximum and absolute minimum of $f(x,y) = 2x^2 - y^2 + 6y$ on the disk $x^2+y^2 \le 16$. [12]

Solution.

For the critical points inside the disk, we have $f_x(x,y) = 4x = 0$ and $f_y(x,y) = 6 - 2y = 0$. [2]

Thus the critical point for f is $(0,3)$. [1]

Also, $f(0,3) = 9$. [1]

If (x,y) is on the boundary of the disk, we have $x^2+y^2 = 16$ and

$f(\pm\sqrt{16-y^2}, y) = 2(16-y^2) - y^2 + 6y = 32 - 3y^2 + 6y$. [1]

Let $g : [-4,4] \longrightarrow \mathbb{R}$ be defined as $g(y) = 32 - 3y^2 + 6y$.

For the critical points of g on $(-4,4)$, we have $g'(y) = -6y + 6 = 0 \Rightarrow y = 1$. [1]

For $y = 1$, we have $x = \pm\sqrt{15}$. [1]

The values of the function f on the boundary of the disk are as follows:

$f(0,4) = 8,\ f(0,-4) = -40$ and $f(\sqrt{15},1) = 35 = f(-\sqrt{15},1)$. [3]

The function f has an absolute minimum at $(0,-4)$ while the absolute maximum occurs twice at $(\sqrt{15},1)$ and $(-\sqrt{15},1)$. [2]