Indian Institute of Information Technology Allahabad

End Semester Examination, December 2016

Date of Examination (Meeting): 10.12.2016 (1st meeting)

Program Code & Semester: B.Tech. (IT), B.Tech. (ECE), Dual Degree - Semester I

Paper Title: Mathematics - I, Paper Code: SMAT130C

Paper Setter: Abdullah Bin Abu Baker & Sumit Kumar Upadhyay

Max Marks: 75 Duration: 3 hours

Attempt **each** question on a new page, and attempt all the parts of a question at the same place. Numbers indicated on the right in [ ] are full marks of that particular problem. All the notations used are standard and same as used in lectures. **Do not write** on question paper and cover pages of the answer booklet except your detail. This question paper has **two** pages. Use of Calculator is **NOT** allowed.

---

1. Provide a short proof or answer of the following statements.
   (a) Let $A, B \in \mathbb{R}$ such that $A \subseteq B$. Then $\inf B \leq \inf A$. [2]
   (b) Show that $\lim_{x \to 0} \cos \frac{1}{x}$ does not exist. [2]
   (c) Find the radius of convergence of series [2]
   $$\frac{1}{2}x + \frac{1.3}{2.5}x^2 + \frac{1.3.5}{2.5.8}x^3 + \cdots .$$
   (d) Find the Macluarin series of the function defined by [3]
   $$f(x) = \begin{cases} e^{-1/x^2}, & \text{for } x \neq 0 \\ 0, & \text{for } x = 0. \end{cases}$$
   (e) Examine the convergence of the integral $\int_1^\infty e^{-x^2}\ dx$. [2]
2. Find the limit of the sequence $(\frac{\sin n}{n}, e^{(\sqrt{n}-\sqrt{n+1})}, \log(1+\frac{1}{n^3}))$ in $\mathbb{R}^3$. [5]
3. (a) Let (x_n) be a bounded sequence. Assume that $x_{n+1} \geq x_n - 2^{-n}$. Show that (x_n) is convergent. [4]
   (b) Let $f : [0,1] \longrightarrow \mathbb{R}$ and $a_n := f(\frac{1}{n}) - f(\frac{1}{n+1})$. Prove the following.
      i. If f is continuous, then $\sum_{n=1}^{\infty} a_n$ converges. [2]
      ii. If f is differentiable and $|f'(x)| < \frac{1}{2},\ \forall\ x \in [0,1]$, then $\sum_{n=1}^{\infty} a_n \sqrt{n} \cos n$ converges. [4]
4. The region bounded by the functions $y = x^2 + x + 1,\ y = 1$ and $x = 1$ is revolved about the line $x = 2$. Find the volume of the solid generated by the shell method. [4]
5. Find the length of the curve [3]
   $$y = \frac{4\sqrt{2}}{3}x^{3/2} - 1, \quad 0 \leq x \leq 1.$$

6. The curve $x(t) = 2\cos t - \cos 2t,\ y(t) = 2\sin t - \sin 2t,\ 0 \le t \le \pi$ is revolved about the x-axis. Calculate the area of the surface generated. [4]

7. Consider the function

$$f(x,y) = \begin{cases} \frac{3x^2y-y^3}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

Answer the following.

(a) Discuss the continuity of f at $(0,0)$. [2]

(b) Evaluate $f_y(x,0)$ for $x \neq 0$. [2]

(c) Is f_y continuous at $(0,0)$. [3]

(d) Find the directional derivative of f at $(0,0)$ in the direction of $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. [2]

(e) Discuss the differentiability of f at $(0,0)$. [3]

8. Evaluate the following integrals:

(a) $\displaystyle\int_0^1 \int_{x^2}^1 x^3 e^{y^3}\, dy\ dx.$ [4]

(b) $\displaystyle\iiint_D \frac{z}{(x^2+y^2+z^2)^{3/2}}\ dx\ dy\ dz$; where D is the region bounded above by the sphere

$x^2 + y^2 + z^2 = 2$ and below by the plane $z = 1$. [7]

9. Let $f(x,y) = (xy^2, x^2y + 2x)$ and C be any square in the plane. Show that the line integral of f along C depends on the area of the square and not on its location in the plane. [3]

10. Find the absolute maximum and absolute minimum of the function $f(x,y) = 2x^2 - y^2 + 6y$ on the disk $x^2 + y^2 \le 16$. [12]